# Comparison of two Metaplectic Cocycles

James Allan Spelling

Ph. D. Thesis
2004

Department of Mathematics
University College London

Supervisor: Dr. Richard Hill.

## Abstract

In my thesis I shall be investigating two distinct metaplectic extensions of the general linear group. The first of these was discovered by Matsumoto, it's existence intimately connected with the deep properties of the r-th order Hilbert symbol. His construction relies heavily on class field theory and algebraic K-theory. Having constructed his metaplectic group, which is known to be universal, Matsumoto was then able to define the cocycle representing this extension.

The second of these metaplectic extensions was found recently by Dr Hill at University College London. In contrast, his construction is very elementary. He was able to prove the existence of a continuous cocycle resulting in the construction of a new non-trivial metaplectic extension. It has already been shown, by Hill, that these two metaplectic extensions are isomorphic if we restrict to the special linear group. However, little is known of this isomorphism.

Throughout this thesis we shall investigate these two cocycles, finding explicit formulae in both cases. We shall then show that the isomorphism between the group extensions of Matsumoto and Hill may be defined via the discovery of the coboundary which splits the quotient of the corresponding cocycles. Having found this coboundary we shall then be able to prove that, in specific cases, the two extensions are in fact isomorphic over the full general linear group.

**Acknowledgments**

I would like to take this opportunity to thank all those who have helped and encouraged me during my years of study at University College London.

I owe a special debt of gratitude to my supervisor, Dr Richard Hill, for all of his guidance and support throughout the research and preparation of this work.

My thanks are extended to the Science and Engineering Research Council for their financial support during my research for this Ph.D.

Finally I would like to dedicate this thesis to my family whose support and encouragement throughout this endevour has been unfaltering.

# Contents

# Chapter 1

# Introduction

In this first chapter we shall discuss some of the background to the problem we are facing. Firstly, we shall look at metaplectic covers and metaplectic extensions. We shall then consider the situation locally, explaining the problem we are facing and the motivation behind this work. The remainder of this chapter will be concerned with defining the notation we shall require and then introducing the metaplectic cocycles with which we shall be working throughout this paper.

## 1.1 Metaplectic Extensions

Let $k$ be a global field, with $\mathbb{A}$ the adèle ring of $k$ and $k_\nu$ the completion of $k$ at the place $\nu$. If we let $G$ be an algebraic group defined over $k$ then $G(\mathbb{A})$ is a topological group with the adèle topology.

We define a metaplectic extension of $G$ to be a topological central extension of $G(\mathbb{A})$ by a finite abelian group $\mu$,

$$1 \to \mu \longrightarrow \tilde{G}(\mathbb{A}) \longrightarrow G(\mathbb{A}) \to 1,$$

such that the group extension splits over $G(k)$.

Thus, metaplectic extensions of $G$ by $\mu$ are classified by elements of $H^2(G(\mathbb{A}), \mu)$ which split when restricted to $G(k)$. The covering group $\tilde{G}(\mathbb{A})$ is called a metaplectic group or a metaplectic cover of $G$.

**note:** Let me point out that some authors have also required that a 2-cocycle $\sigma_{\mathbb{A}}$ corresponding to a metaplectic extension may be decomposed as,

$$\sigma_{\mathbb{A}}(\alpha, \beta) = \prod_\nu \sigma_\nu(\alpha_\nu, \beta_\nu),$$

where $\alpha_\nu$, $\beta_\nu$ are the components of $\alpha$ and $\beta$ in $G(k_\nu)$ and the product is taken over all places of $k$ with $\sigma_\nu$ being a 2-cocycle on $G(k_\nu)$. However, not all Metaplectic 2-cocycles have this property. That is, the metaplectic cocycle Dec, and corresponding metaplectic extension found by Hill, does not satisfy this condition. Instead, his cocycle results in a slightly different expression (Thm 4, [3]).

If $k$ contains some primitive $q^{\text{th}}$ root of unity then there exists a canonical metaplectic extension of the group $\mathrm{SL}_n/k$ by the finite abelian group $\mu_q$ of all $q^{\text{th}}$ roots in $k$. This extension was first constructed by T. Kubota in [5], [6] in the case $n = 2$ and by H. Matsumoto [7] for general $n$. This extension is always non-trivial and if $n \geq 3$ and $q$ is the total number of roots of unity in $k$ then this extension is known to be universal amongst metaplectic extensions. Since $\mathrm{GL}_n(\mathbb{A})$ and $\mathrm{GL}_n(k_\nu)$ are not perfect, the group $\mathrm{GL}_n$ has no universal metaplectic extension. However, a metaplectic extension of $\mathrm{GL}_n/k$ with values in $\mu_q$ may be constructed by embedding $\mathrm{GL}_n$ in $\mathrm{SL}_{n+1}$ in a particular way.

Given a metaplectic extension,

$$1 \to \mu \longrightarrow \tilde{G}(\mathbb{A}) \longrightarrow G(\mathbb{A}) \to 1,$$

and corresponding cocycle $\sigma_{\mathbb{A}}$ we are able to explicitly realize the metaplectic group $\tilde{G}(\mathbb{A})$ as the set of pairs,

$$\tilde{G}(\mathbb{A}) = \{(g, \xi) : g \in G(\mathbb{A}),\ \xi \in \mu\},$$

where multiplication in the group is defined such that,

$$(g_1, \xi_1)\cdot(g_2, \xi_2) = (g_1 g_2, \xi_1 \xi_2 \sigma_{\mathbb{A}}(g_1, g_2)).$$

Let us also note that, given any central extension, we may recover a corresponding 2-cocycle $\sigma_{\mathbb{A}}$ by,

$$\sigma_{\mathbb{A}}(g, h) = \hat{g}\ \hat{h}\ (\hat{gh})^{-1},$$

where for each $g \in G(\mathbb{A})$ we choose a preimage $\hat{g}$ in $\tilde{G}(\mathbb{A})$.

**Example**

Let $k$ be a global field containing $\mu_q$ a group of roots of unity with $q$ elements. For each $\alpha$, $\beta \in \mathrm{GL}_1(\mathbb{A})$, we define

$$(\alpha, \beta)_{\mathbb{A},q} = \prod_{\nu} (\alpha_\nu, \beta_\nu)_{\nu,q},$$

where $(-,-)_{\nu,q}$ is the $q^{\text{th}}$ power Hilbert symbol on $k_\nu$ for each place $\nu$.

Clearly this is a 2-cocycle and the power reciprocity law for $k$ ensures that it splits over $\mathrm{GL}_1(k)$. Therefore this cocycle corresponds to,

$$1 \to \mu_q \longrightarrow \tilde{\mathrm{GL}}_1(\mathbb{A}) \longrightarrow \mathrm{GL}_1(\mathbb{A}) \to 1,$$

a metaplectic cover of $\mathrm{GL}_1$ by $\mu_q$.

### Local Metaplectic Extensions

Let us now consider the situation locally. Let,

$$1 \to \mu \longrightarrow \tilde{\mathrm{GL}}_n(\mathbb{A}) \xrightarrow{\varrho} \mathrm{GL}_n(\mathbb{A}) \to 1,$$

be some metaplectic extension of $\mathrm{GL}_n(\mathbb{A})$ by a finite abelian group $\mu$ corresponding to the 2-cocycle $\sigma_{\mathbb{A}}$. Then, having defined the 2-cocycle $\sigma_\nu$ by,

$$\sigma_\nu = \sigma_{\mathbb{A}}|_{\mathrm{GL}_n(k_\nu)} ,$$

it is clear that at the place $\nu$ we shall have a local metaplectic cover,

$$1 \to \mu \longrightarrow \tilde{\mathrm{GL}}_n(k_\nu) \longrightarrow \mathrm{GL}_n(k_\nu) \to 1,$$

where the local metaplectic group $\tilde{\mathrm{GL}}_n(k_\nu)$ is taken to be the pre-image $\varrho^{-1}(\mathrm{GL}_n(k_\nu))$ in $\tilde{\mathrm{GL}}_n(\mathbb{A})$. This group may once again be realized by,

$$\tilde{\mathrm{GL}}_n(k_\nu) = \{(g, \xi) : g \in \mathrm{GL}_n(k_\nu),\ \xi \in \mu\},$$

with multiplication,

$$(g_1, \xi_1).(g_2, \xi_2) = (g_1 g_2, \xi_1 \xi_2 \sigma_\nu(g_1, g_2)).$$

It is with these local metaplectic covers that our work in this thesis is concerned.

## 1.1.1 The Aim of this Thesis

At the place $\nu$ it is known that there exist two realizations of the local metaplectic cover,

$$\begin{aligned} 1 \to \mu_m \longrightarrow \tilde{\mathcal{G}}_\sigma \quad &\longrightarrow \mathrm{GL}_n(k_\nu) \to 1 \\ 1 \to \mu_m \longrightarrow \tilde{\mathcal{G}}_{dec_\nu} &\longrightarrow \mathrm{GL}_n(k_\nu) \to 1, \end{aligned}$$

represented by two distinct cocycles $\sigma_n$ and $dec_\nu$ in $Z^2(\mathrm{GL}_n(k_\nu), \mu_m)$, where $\mu_m \subset k_\nu$ is a set of $m^{th}$ roots of unity.

As we have seen we may realize each of these groups as the set of pairs $\mathrm{GL}_n(k_\nu) \times \mu_m$ such that,

$$\tilde{\mathcal{G}}_\sigma = \langle (g,\xi) : (g,\xi).(g',\xi') = (gg', \xi\xi'\sigma_n(g,g')) \rangle$$
$$\tilde{\mathcal{G}}_{dec_\nu} = \langle (g,\xi) : (g,\xi).(g',\xi') = (gg', \xi\xi'\, dec_\nu(g,g')) \rangle.$$

If we restrict our extensions to $\mathrm{SL}_n$, the metaplectic groups $\tilde{\mathcal{G}}_\sigma$ and $\tilde{\mathcal{G}}_{dec_\nu}$ have been proven to be isomorphic (Thm 6 [3]). Throughout the course of this thesis we shall endevour to explicitly define this isomorphism. That is, we know that

$$\tilde{\mathcal{G}}_\sigma \cong |_{\mathrm{SL}_n} \tilde{\mathcal{G}}_{dec_\nu}, \qquad \text{given by,} \quad (g,\xi) \longmapsto (g, \xi\psi(g)).$$

By considering the expression of $\tilde{\mathcal{G}}_\sigma$ and $\tilde{\mathcal{G}}_{dec_\nu}$ as the set of pairs $\mathrm{GL}_n(k_\nu) \times \mu_m$, it should be clear that the map $\psi$ is precisely the cochain which splits the quotient of the cocycles $\sigma_n$ and $dec_\nu$ over $\mathrm{SL}_n$. That is,

$$\partial\psi = \frac{dec_\nu}{\sigma_n}.$$

In the work that follows, if $m$, the number of roots of unity in the group $\mu_m$, is odd we shall in fact be able to go a little further than this. By slightly changing our original definition of the cocycle $\sigma_n$ we are able to show that we have,

$$\tilde{\mathcal{G}}_\sigma \cong |_{\mathrm{GL}_n} \tilde{\mathcal{G}}_{dec_\nu}, \qquad \text{given by,} \quad (g,\xi) \longmapsto (g, \xi\psi(g)),$$

and are once again able to explicitly find the isomorphism.

### 1.1.2 Structure of this Thesis

**Chapter 1:** We begin by fixing some standard algebraic notation. We then discuss the two, very different constructions of the cocycles $\sigma_n$ and $dec_\nu$ which result in different metaplectic extensions of $\mathrm{GL}_n(k_\nu)$.

**Chapter 2:** We concentrate on finding some algebraic expression to describe the cocycle $dec_\nu$ when restricted to the torus of diagonal matrices in $\mathrm{GL}_n(k_\nu)$. In the case that $m$, the number of roots of unity in $\mu_m$, is even we shall find that we are only able to calculate the value of this cocycle $dec_\nu$ up to some coboundary $\partial\tau_n$.

**Chapter 3:** We return to the case when $m$ is even and calculate the value of the coboundary $\partial\tau_n$ associated with $dec_\nu$ on the torus $T$ in $\mathrm{GL}_n(k_\nu)$.

**Chapter 4:** We calculate an expression for the cocycle $dec_\nu$ when restricted to the monomial matrices in $\mathrm{GL}_n(k_\nu)$. Since the cocycle $\sigma_n$ is defined differently to $dec_\nu$ we shall in fact consider the monomials to be defined in two different ways, calculating results for $dec_\nu$ in both cases.

**Chapter 5:** We consider the value of the cocycle $dec_\nu$ on the full general linear group $\mathrm{GL}_2(k_\nu)$. Although we shall not be able to describe the value of the cocycle in full we are, however, able to find enough results to fully describe the splitting of the quotient $dec_\nu/\sigma_2$ in Chapter 9.

**Chapter 6:** We concentrate on finding results for the cocycle $dec_\nu$ when restricted to $N$, the subgroup of upper triangular matrices in $\mathrm{GL}_n(k_\nu)$.

**Chapter 7:** We return to the definition of the Matsumoto's cocycle $\sigma_n$. Extending some known results for the cocycle, found in [1], we are able to find expressions for the value of $\sigma_n$ on $\mathrm{GL}_n(k_\nu)$ in much the same way as we did for $dec_\nu$.

**Chapter 8:** Having found formulae for both cocycles we consider the splitting of the quotient $dec_\nu/\sigma_n = \partial\psi$ and the isomorphism between the related metaplectic extensions of $\mathrm{GL}_n(k_\nu)$. Concentrating on the groups $T$, $M$ and $N$ in turn we are able to go a long way to describing the function $\psi$ on $\mathrm{GL}_n(k_\nu)$.

**Chapter 9:** In the final chapter we shall turn our attention entirely towards $\mathrm{GL}_2(k_\nu)$ and explicitly describe the function $\psi$ and the isomorphism between the metaplectic groups relating to $dec_\nu$ and $\sigma_2$.

## 1.2 Notation

### 1.2.1 The Local Field $k_\nu$

Throughout this paper we let $k_\nu$ be the completion of the global field $k$ at some finite place $\nu$ and define $\mathfrak{O}_\nu \subset k_\nu$ to be the ring of integers. Having defined $\wp_\nu$ to be the unique maximal ideal in $\mathfrak{O}_\nu$ we choose and fix a uniformizing element $\pi \in \wp_\nu \setminus \wp_\nu^2$.

We have an absolute value on $k_\nu$ normalized such that,

$$|\alpha|_\nu = \frac{1}{|(\mathfrak{O}_\nu/\alpha\mathfrak{O}_\nu)|}, \qquad \text{for each } \alpha \in \mathfrak{O}_\nu.$$

Then, for our fixed uniformizer $\pi \in \mathfrak{O}_\nu$, we have

$$|\pi|_\nu = \frac{1}{p^f} =: \rho^{-1},$$

where $f = [\mathfrak{O}_\nu/\wp_\nu : \mathbb{F}_p]$ is the degree of the extension.

### 1.2.2 The Finite Abelian Group $\mu_q$

Let $\mu_q = (\mathfrak{O}_\nu/\wp_\nu)^\times$. Then, $\mu_q$ is the group of all roots of unity in $k_\nu$ with order coprime to $p$. Thus, we may write

$$\mu_q = \mu_{2^r} \oplus \mu_t,$$

where $q = \rho - 1 = 2^r t$ with $t$ odd.

Let $(-,-)_{\nu,q}$ be the $q^{\text{th}}$ power Hilbert symbol on $k_\nu$, for each place $\nu$,

$$(-,-)_{\nu,q} : k_\nu{}^\times \times \; k_\nu{}^\times \longrightarrow \mu_q.$$

Using the Chinese remainder theorem we find that, for each $a,\ b \in k_\nu$, there exists integers $\Lambda_1, \Lambda_2 \in \mathbb{Z}$ such that,

$$(a,b)_{\nu,q} = (a,b)_{\nu,2^r}^{\Lambda_1}.(a,b)_{\nu,t}^{\Lambda_2}.$$

The Hilbert symbol also satisfies,

$$(a,b)_{\nu,2^r} = (a,b)_{\nu,q}^{t} \qquad\qquad (a,b)_{\nu,t} = (a,b)_{\nu,q}^{2^r}.$$

Since all of our cocycles are to be expressed as Hilbert symbols and since we already know that,

$$H^2(\mathrm{GL}_n(k_\nu), \mu_q) = H^2(\mathrm{GL}_n(k_\nu), \mu_{2^r}) \oplus H^2(\mathrm{GL}_n(k_\nu), \mu_t),$$

it shall be sufficient for us to consider local metaplectic extensions of $\mathrm{GL}_n/\,k_\nu$ with values in the group of roots of unity $\mu_m$ where we have either $m = 2^r$ or $m = t$.

In order to calculate the 2-cocycles in both cases $m = 2^r$ and $m = t$ simultaneously, it shall be convenient for us to re-define the symbol $(-1)$. That is, for the remainder of this thesis we shall define,

$$(-1) := \prod_{\xi \in \mu_m} \xi.$$

Then, clearly we shall have $(-1) = 1$ is trivial, whenever $m = t$ is odd.

### 1.2.3 The General Linear Group $\mathrm{GL}_n(k_\nu)$

#### The Torus $T \subset \mathrm{GL}_n(k_\nu)$

Let $T \subset \mathrm{GL}_n(k_\nu)$ be the torus of diagonal matrices with $\mathbb{T} = T \cap \mathrm{SL}_n(k_\nu)$. Also let us define $T_{\mathbf{Z}} \subset \mathbb{T}$ to be the subgroup of matrices with entries $\pm 1$. Throughout this work we shall define, $\alpha_n,\ \beta_n \in T$ by,

$$\alpha_n = \begin{pmatrix} \pi^{X_1} a_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \pi^{X_n} a_n \end{pmatrix}, \qquad \beta_n = \begin{pmatrix} \pi^{Y_1} b_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \pi^{Y_n} b_n \end{pmatrix}.$$

Let $\tau_i : T \mapsto k_\nu$ to be the map which picks out the $i^{th}$ diagonal entry. Let us also define $\Phi$ to be the set of roots of $\mathrm{GL}_n(k_\nu)$ relative to $T$, given by

$$\Phi = \{(i,j) : 1 \le i,\, j \le n, i \ne j\}.$$

Having defined $\Phi$ relative to $T$, for each $\varsigma = (i,j) \in \Phi$, we have

$$\alpha_n^\varsigma = \tau_i(\alpha_n)/\tau_j(\alpha_n) = \pi^{X_i}a_i/\pi^{X_j}a_j.$$

The root $(i,j)$ is said to be positive if $i < j$. We define $\Phi^+ = \{\varsigma \in \Phi : i < j\}$ to be the set of all positive roots with the ordered base of simple roots,

$$\Delta = \{\varsigma = (i, i+1) \in \Phi : 1 \le i < n\}.$$

The set $\Phi^-$ is similarly defined.

### The Monomials $M \subset \mathrm{GL}_n(k_\nu)$

Let $M \subset \mathrm{GL}_n(k_\nu)$ be the group of monomial matrices with $\mathbb{M}$ being the restriction to $\mathrm{SL}_n(k_\nu)$.

We have an action of $M$ on $T$ by conjugation where, for each $\alpha_n \in T$, $m \in M$ we have

$$\alpha_n^m := m^{-1}\alpha_n m \qquad \text{and} \qquad {}^m\alpha_n := m\alpha_n m^{-1} = \alpha_n^{m^{-1}}.$$

This action of $M$ on $T$ induces an action on $\Phi$. That is, $m\varsigma$ is the unique element of $\Phi$ such that

$$\alpha_n^{(m\varsigma)} = (\alpha_n^m)^\varsigma \qquad \text{for all } \alpha_n \in T.$$

The group $M$ also acts on the set of integers $\{1, \ldots, n\}$ with $mi$ the unique element of the set satisfying,

$$\tau_{mi}(\alpha_n) = \tau_i(\alpha_n^m) \qquad \text{for all } \alpha_n \in T.$$

As a consequence of this we have the relation,

$$m(i,j) = (mi, mj) \qquad \text{for all } m \in M, (i,j) \in \Phi.$$

Let us now define $W \subset M$ to be the group of permutation matrices. That is the group of matrices such that each row or column has just one non-zero entry which is a 1. Then clearly we have $M = T.W$.

Since, when we restrict to $\mathbb{M}$, we still have an action on $\Phi$ with $\mathbb{T}$ acting trivially we have a well defined faithful action of $W \cong \mathbb{M}/\mathbb{T}$ on $\Phi$. Thus we may regard each $w \in W$

as a permutation on $\Phi$. Furthermore we define the *weight space* of $w$, $\Phi(w)$, to be the set of all positive roots which, when acted upon by $w$, are no longer positive. That is,

$$\begin{aligned}\Phi(w) &= \{\varsigma \in \Phi^+ : w\varsigma \notin \Phi^+\} \\ &= \{(i,j) \in \Phi : i < j,\ w(i) > w(j)\}.\end{aligned}$$

Since we have,

$$W \cong S_n = \langle (i, i+1) : 1 \leq i < n \rangle,$$

the group $W$ is generated by the simple reflections $\{s_\varsigma = s_{(i,i+1)} : \varsigma = (i, i+1) \in \Delta\}$. Therefore each $w \in W$ may be expressed as,

$$w = s_{\varsigma_1} \dots s_{\varsigma_\ell}, \quad \text{where } \varsigma_k = (i_k, i_k + 1) \in \Delta. \tag{1.1}$$

We now define the length of $w$, $l(w)$, to be the length of the shortest expression of $w$ as a product of the $s_\varsigma$'s, with $l(1) = 0$. If $l(w) = \ell$, then any expression of the form (1.1) is called a *reduced expression* for $w$.

Let $\mathrm{M}_{\mathbf{Z}}$ be the monomials generated by the elements,

$$\{w_\varsigma : \varsigma = (i, i+1) \in \Delta\},$$

where $w_\varsigma$ is the monomial matrix with a -1 in the $(i,j)$-position, a 1 in the $(j,i)$-position and ones along the diagonal.

Since $W \not\subset \mathrm{SL}$ and the construction of Matsumoto's cocycle relies on the construction of the metaplectic cocycle on $\mathrm{SL}_n$, we shall have to define a new set $\mathfrak{M}$ as follows.

For each $w \in W$ we define,

$$\eta_w = w_{\varsigma_1} \dots w_{\varsigma_\ell},$$

where $w = s_{\varsigma_1} \dots s_{\varsigma_\ell}$ is some reduced expression for $w$ with $l(w) = \ell$. Then $\eta_w \in \mathrm{M}_{\mathbf{Z}}$ and $w \mapsto \eta_w$ is independent of the expression of $w$ as a product of $l(w)$ simple reflections. We now define the set

$$\mathfrak{M} = \{\eta_w : w \in W\}.$$

Then, although this isn't a group in general, the map $w \mapsto \eta_w$ gives a bijection $W \xrightarrow{\sim} \mathfrak{M}$ and $\mathfrak{M}$ is a complete set of coset representatives for $\mathbb{M}/\mathbb{T}$.

### The Group of Unipotent Upper-triangular Matrices $N \subset \mathrm{GL}_n(k_\nu)$

Our choices of the sets $\Phi$ and $\Delta$ in the previous section determine a Borel subgroup of $\mathrm{GL}_n(k_\nu)$ whose unipotent radical we denote by $N$. For each $\varsigma \in \Phi$ we also let $\mathrm{N}_\varsigma$ be the standard unipotent subgroup of $\mathrm{SL}_n(k_\nu)$ corresponding to $\varsigma$.

It is well known that the subgroups $\{\mathrm{N}_\varsigma : \varsigma \in \Phi\}$ generate $\mathrm{SL}_n(k_\nu)$ as an abstract group. That is, the group $\mathrm{SL}_n(k_\nu)$ may be generated by the following:

1. For every $\varsigma \in \Phi^+, a \in k_\nu$:
$$n_\varsigma(a) = I + a.e_\varsigma,$$
(which is simply the matrix with an $a$ in the $\varsigma^{th}$-position and ones on the diagonal.)

2. For every $\varsigma \in \Delta$:
$$w_\varsigma = n_\varsigma(-1)n_{-\varsigma}(1)n_\varsigma(-1).$$
(which is the monomial matrix with a -1 in the $\varsigma$-position, a 1 in the $-\varsigma$-position and ones along the diagonal.)

3. For every $\varsigma \in \Delta, a \in {k_\nu}^\times$:
$$\alpha_\varsigma(a) = n_\varsigma(a)n_{-\varsigma}(-a^{-1})n_\varsigma(a)w_\varsigma.$$
(which is the diagonal matrix with $a$ in the $i^{th}$-position, $a^{-1}$ in the $(i+1)^{th}$-position and ones elsewhere.)

Let us also note that, for each $\varsigma \in \Phi$, we may consider $\mathrm{N}_\varsigma$ to be the image of the isomorphism,
$$n_\varsigma : k_\nu \longrightarrow \mathrm{SL}_n(k_\nu),$$
as described above in 1.

Let us recall that having defined $\mathfrak{M} = \{\eta_w : w \in W\}$, the map $w \mapsto \eta_w$ gives a bijection $W \tilde{\rightarrow} \mathfrak{M}$ and $\mathfrak{M}$ is a complete set of coset representatives for $W \cong \mathbb{M}/\mathbb{T}$.

Then, for $\mathrm{GL}_n(k_\nu)$, we have the Bruhat decomposition,
$$\mathrm{GL}_n(k_\nu) = \coprod_{\eta_w \in \mathfrak{M}} NT\eta_w N.$$

### 1.2.4 2-cocycles on $\mathrm{GL}_n(k_\nu)$

A 2-cocycle on $\mathrm{GL}_n(k_\nu)$ with coefficients in $\mu_m$ is a map $\sigma : \mathrm{GL}_n(k_\nu) \times \mathrm{GL}_n(k_\nu) \longrightarrow \mu_m$, such that

$$\begin{aligned} &\sigma(g_1g_2, g_3)\sigma(g_1, g_2) = \sigma(g_1, g_2g_3)\sigma(g_2, g_3), && g_1, g_2, g_3 \in \mathrm{GL}_n(k_\nu), \\ &\sigma(I_n, I_n) = \sigma(I_n, g) = \sigma(g, I_n) = 1 && \text{for all } g \in \mathrm{GL}_n(k_\nu), \end{aligned}$$

and where $I_n$ is the identity in $\mathrm{GL}_n(k_\nu)$. We let $Z^2(\mathrm{GL}_n(k_\nu), \mu_m)$ denote the set of all such 2-cocycles.

## 1.3 Matsumoto's cocycle $\sigma$

In this section we shall consider the construction of an explicit metaplectic 2-cocycle $\sigma_{\mathrm{SL}} \in Z^2(\mathrm{SL}_n(k_\nu), \mu_m)$ representing the cohomology class in $H^2(\mathrm{SL}_n(k_\nu), \mu_m)$ of the universal metaplectic extension of $\mathrm{SL}_n(k_\nu)$ by $\mu_m$. Although this construction was first discovered by Matsumoto in [7], the main reference for the results in this section is the work of Banks, Levy and Sepanski [1].

Furthermore, in [1], by pulling back $\sigma_{\mathrm{SL}}$ from $\mathrm{SL}_{n+1}(k_\nu)$ to $\mathrm{GL}_n(k_\nu)$ under a particular embedding, they obtained for every $n \geq 1$ an explicit 2-cocycle $\sigma_n \in Z^2(\mathrm{GL}_n(k_\nu), \mu_m)$ that represents the second cohomology class of the central extension $\tilde{\mathrm{GL}}_n(k_\nu)$ of $\mathrm{GL}_n(k_\nu)$ by $\mu_m$, where $\tilde{\mathrm{GL}}_n(k_\nu)$ is the pre-image of $\mathrm{GL}_n(k_\nu)$ in $\tilde{\mathrm{SL}}_{n+1}(k_\nu)$. They were also able to show that the cocycle $\sigma_2$ is identical to the cocycle $\sigma_k$ described by Kubota [5], [6]. We shall now give a brief description of the method they employed.

### 1.3.1 Construction of the cocycle

#### The projection $\varrho$

The metaplectic central extension of $\mathrm{SL}_n(k_\nu)$ by $\mu_m$ is constructed via the realization of a natural projection $\varrho : \tilde{\mathrm{SL}}_n(k_\nu) \longrightarrow \mathrm{SL}_n(k_\nu)$, giving an exact sequence,

$$1 \rightarrow \mu_m \hookrightarrow \tilde{\mathrm{SL}}_n(k_\nu) \xrightarrow{\varrho} \mathrm{SL}_n(k_\nu) \rightarrow 1,$$

In [8] Steinberg showed that there exist unique lifts,

$$\{n_\varsigma^* : k_\nu \mapsto \tilde{\mathrm{SL}}_n(k_\nu) : \varsigma \in \Phi\}$$

of the maps $\{n_\varsigma : \varsigma \in \Phi\}$ given earlier, satisfying the same set of relations. In particular, Steinberg has shown that the projection $\varrho$ induces an isomorphism from the subgroup $N^*$ of $\tilde{\mathrm{SL}}_n(k_\nu)$ to the subgroup $N$ of $\mathrm{SL}_n(k_\nu)$.

For each $\varsigma \in \Phi, a \in {k_\nu}^\times$, we define the elements of the metaplectic group

$$\tilde{w}_\varsigma(a) = n_\varsigma^*(a) n_{-\varsigma}^*(-a^{-1}) n_\varsigma^*(a), \qquad \tilde{\alpha}_\varsigma(a) = \tilde{w}_\varsigma(a)\tilde{w}_\varsigma(1)^{-1}.$$

It has also been shown by Steinberg (See [1]) that there exists a presentation of $\tilde{\mathrm{SL}}_n(k_\nu)$ where the set of generators is the union of the sets,

$$\mu_m, \quad \{\tilde{\alpha}_\varsigma(a) : \varsigma \in \Delta, a \in {k_\nu}^\times\}, \qquad \{\tilde{w}_\varsigma : \varsigma \in \Delta\}, \qquad \{n_\varsigma^*(a) : \varsigma \in \Phi^+, a \in k_\nu\},$$

with $\tilde{w}_\varsigma := \tilde{w}_\varsigma(-1)$ for all $\varsigma \in \Phi$.

Using this presentation, we may now describe the natural homomorphism

$$\varrho : \tilde{\mathrm{SL}}_n(k_\nu) \longmapsto \mathrm{SL}_n(k_\nu)$$

by its values on the generators given above. That is,

$$\begin{array}{lll} \varrho(\xi) & = I_n, & \xi \in \mu_m, \\ \varrho(\tilde{\alpha}_\varsigma(a)) & = \alpha_\varsigma(a), & \varsigma \in \Delta,\ a \in {k_\nu}^\times, \\ \varrho(\tilde{w}_\varsigma) & = w_\varsigma, & \varsigma \in \Delta, \\ \varrho(n_\varsigma^*(a)) & = n_\varsigma(a), & \varsigma \in \Phi^+,\ a \in k_\nu . \end{array}$$

### The section $\mathfrak{s}$

In order to define the cocycle we must now look for a *section* of $\varrho$ on $\mathrm{SL}_n(k_\nu)$, a map $\mathfrak{s} : \mathrm{SL}_n(k_\nu) \to \tilde{\mathrm{SL}}_n(k_\nu)$, such that

$$\varrho(\mathfrak{s}(g)) = g \qquad \text{and} \quad \mathfrak{s}(I_n) = 1.$$

When Steinberg was able to show that $\varrho : N^* \longmapsto N$ is an isomorphism, he proved that there exists a section $\mathfrak{s}_N$ of $N$ with the property,

$$\mathfrak{s}_N(n_\varsigma(a)) = n_\varsigma^*(a) \qquad \forall \varsigma \in \Phi^+, a \in k_\nu.$$

We now define a section $\mathfrak{s}_{\mathbb{T}}$ on $\mathbb{T}$ as follows,

$$\mathfrak{s}_{\mathbb{T}}(\alpha_\varsigma(a)) := \tilde{\alpha_\varsigma}(a).(a,a)_{\nu,m} \qquad \forall \varsigma \in \Delta, a \in {k_\nu}^\times.$$

It is easily shown that all $\alpha \in \mathbb{T}$ may be uniquely expressed as a product,

$$\alpha = \prod_1^{n-1} \alpha_{\varsigma_i}(a_i), \qquad \text{where } \varsigma_k = (i, i+1).$$

Therefore, for general elements of the group $\mathbb{T}$ we define,

$$\mathfrak{s}_{\mathbb{T}}(\alpha) = \mathfrak{s}_{\mathbb{T}}(\alpha_{\varsigma_{n-1}}(a_{n-1}))\dots\mathfrak{s}_{\mathbb{T}}(\alpha_{\varsigma_1}(a_1)).$$

We are now able to extend the above section from $\mathbb{T}$ to a unique section $\mathfrak{s}_{\mathbb{M}}$ on $\mathbb{M}$ by defining,

$$\begin{array}{rl} \mathfrak{s}_{\mathbb{M}}(w_\varsigma) = \tilde{w_\varsigma} & \forall \varsigma \in \Delta \\ \mathfrak{s}_{\mathbb{M}}(\eta_1\eta_2) = \mathfrak{s}_{\mathbb{M}}(\eta_1)\mathfrak{s}_{\mathbb{M}}(\eta_2) & \forall \eta_1, \eta_2 \in \mathfrak{M} \text{ such that } l(\eta_1\eta_2) = l(\eta_1) + l(\eta_2) \\ \mathfrak{s}_{\mathbb{M}}(\alpha\eta) = \mathfrak{s}_{\mathbb{T}}(\alpha)\mathfrak{s}_{\mathbb{M}}(\eta) & \forall \alpha \in \mathbb{T}, \eta \in \mathfrak{M}. \end{array}$$

Having defined all of the sections $\mathfrak{s}_N$, $\mathfrak{s}_{\mathbb{T}}$, $\mathfrak{s}_{\mathbb{M}}$ and considering the Bruhat decomposition described earlier we are now able to define a section on the whole of $\mathrm{SL}_n(k_\nu)$ as follows:

$$\mathfrak{s}_{SL_n}(n_1 m n_2) = \mathfrak{s}_N(n_1)\mathfrak{s}_{\mathbb{M}}(m)\mathfrak{s}_N(n_2),$$

for all $n_1, n_2 \in N$ and $m \in \mathbb{M}$.

**Definition:**

We are now in a position to define the cocycle $\sigma_{SL_n} \in Z^2(\mathrm{SL}_n(k_\nu), \mu_m)$ by

$$\sigma_{SL_n}(g_1, g_2) = \frac{\mathfrak{s}_{SL_n}(g_1)\mathfrak{s}_{SL_n}(g_2)}{\mathfrak{s}_{SL_n}(g_1 g_2)} \qquad \text{for all } g_1, g_2 \in \mathrm{SL}_n(k_\nu).$$

From the construction of our section $\mathfrak{s}_{SL_n}$, we immediately have the following relations:

$$\begin{array}{lll}
\sigma_{SL_n}(g, n) & = \sigma_{SL_n}(n, g) = 1 & \forall\, n \in N,\ g \in \mathrm{SL}_n(k_\nu) \\
\sigma_{SL_n}(n_1 g_1, g_2 n_2) & = \sigma_{SL_n}(g_1, g_2) & \forall\, n_1,\ n_2 \in N,\ g_1,\ g_2 \in \mathrm{SL}_n(k_\nu) \\
\sigma_{SL_n}(g_1 n, g_2) & = \sigma_{SL_n}(g_1, n g_2) & \forall\, n \in N,\ g_1,\ g_2 \in \mathrm{SL}_n(k_\nu) \\
\sigma_{SL_n}(\alpha, \eta) & = 1 & \forall\, \alpha \in \mathbb{T},\ \eta \in \mathfrak{M} \\
\sigma_{SL_n}(\eta_1, \eta_2) & = 1 & \forall\, \eta_1,\ \eta_2 \in \mathfrak{M}\ :\ l(\eta_1\eta_2) = l(\eta_1) + l(\eta_2).
\end{array}$$

### Defining the cocycle $\sigma_n$ on $\mathrm{GL}_n$

Finally, by considering the embedding $\imath : \mathrm{GL}_n(k_\nu) \hookrightarrow \mathrm{SL}_{n+1}(k_\nu)$ given by,

$$g \longmapsto \begin{pmatrix} g & 0 \\ 0 & \det(g)^{-1} \end{pmatrix},$$

we are eventually able to define the cocycle $\sigma_n \in Z^2(\mathrm{GL}_n(k_\nu), \mu_m)$ corresponding to,

$$1 \to \mu_m \longrightarrow \tilde{\mathcal{G}_\sigma} \longrightarrow \mathrm{GL}_n(k_\nu) \to 1,$$

the metaplectic cover of $\mathrm{GL}_n(k_\nu)$ by,

$$\sigma_n(g_1, g_2) = \sigma_{SL_{n+1}}(\imath(g_1), \imath(g_2)).(\det(g_1), \det(g_2))_{\nu,m}^{-1}.$$

We shall return to the definition of this cocycle and its properties later in Chapter 7.

## 1.4 Hill's cocycle Dec

Hill's approach in constructing the metaplectic extension of $\mathrm{GL}_n/k_\nu$ is very different to that of Matsumoto. He explicitly constructs $\mathrm{Dec} \in Z^2(\mathrm{GL}_n(k_\nu), \mu_m)$, a continuous metaplectic 2-cocycle corresponding to the extension described earlier. We shall, using [3], briefly describe the ingenious method he employed.

### 1.4.1 Construction of the cocycle

For the remainder of this paper we shall, when convenient, refer to the general linear group as $\mathrm{GL}_n(k_\nu) =: G_\nu$.

#### The Cup Product

Suppose that $\mathfrak{F}$, $\mathfrak{C}$ and $\mu_m$ are all right $G_\nu$-modules. Let us also suppose that we have a bilinear form $<-|->: \mathfrak{F} \times \mathfrak{C} \longrightarrow \mu_m$ which satisfies,

$$< f\alpha | M\alpha > = < f|M >^\alpha, \qquad \forall f \in \mathfrak{F},\ M \in \mathfrak{C} \text{ and } \alpha \in \mathrm{G}_\nu.$$

Then there exists a *cup product* map,

$$\sqcup : H^n(G_\nu, \mathfrak{F}) \times H^m(G_\nu, \mathfrak{C}) \longrightarrow H^{n+m}(G_\nu, \mu_m),$$

such that, if $\kappa_1$ and $\kappa_2$ are 1-cocycles, their cup product is defined to be,

$$(\kappa_1 \sqcup \kappa_2)(\alpha, \beta) := < \kappa_1(\alpha) | \kappa_2(\beta^{-1}) >^\beta .$$

#### The $\mathfrak{C}$-modules

We define $\mathfrak{C}$ to be the abelian group of all functions,

$$M : {k_\nu}^n/\mu_m \longrightarrow \mathbb{Z},$$

which are continuous with compact support. We are able to give $\mathfrak{C}$ a $G_\nu$-module structure by defining,

$$M\alpha(\phi) := M(\alpha\phi) \qquad \text{for all} \quad \alpha \in G_\nu.$$

As such, we have a surjective $G_\nu$-module homomorphism,

$$\deg : \mathfrak{C} \longrightarrow \mathbb{Z}, \qquad \text{where} \quad \deg(M) := M(0).$$

By letting $\mathfrak{C}^0$ be the kernel of the degree map we are able to construct a short exact sequence of $G_\nu$-modules,

$$0 \longrightarrow \mathfrak{C}^0 \longrightarrow \mathfrak{C} \longrightarrow \mathbb{Z} \longrightarrow 0.$$

This, in turn, gives us a long exact sequence of homology groups. Within this sequence we have the map,

$$\delta_{\mathfrak{C}} : H^0(G_\nu, \mathbb{Z}) \longrightarrow H^1(G_\nu, \mathfrak{C}^0).$$

Since $G_\nu$ acts trivially on $\mathbb{Z}$ we have $H^0(G_\nu, \mathbb{Z}) = \mathbb{Z}$, which allows us to describe a canonical element $\delta_{\mathfrak{C}}(1) \in H^1(G_\nu, \mathfrak{C}^0)$.

Finally, by letting $M \in \mathfrak{C}$ have degree 1 and defining,

$$\delta M(\alpha) := M - M\alpha, \qquad \text{for each} \quad \alpha \in G_\nu,$$

we find that $\delta M$ is a continuous 1-cocycle representing the class $\delta_{\mathfrak{C}}(1)$.

**Remark:**

We should point out that any $M \in \mathfrak{C}$ induces a $\mu_m$-invariant function, $M : k_\nu{}^n \longrightarrow \mathbb{Z}$, which we shall also refer to as $M$.

### The $\mathfrak{F}$-modules

We define $\mathfrak{F}$ to be the group of all functions,

$$f : k_\nu{}^n \setminus 0 \longrightarrow \mathbb{Z}, \qquad \text{which satisfy,} \quad \sum_{\xi \in \mu_m} f\xi =: \deg(f)$$

is a constant function on $k_\nu{}^n \setminus 0$. We refer to elements of $\mathfrak{F}$ of degree 1 as *fundamental functions.*

For each $f \in \mathfrak{F}$ and each $\alpha \in G_\nu$ the composition of $f$ with $\alpha$, which we write as $f\alpha$, is also an element of $\mathfrak{F}$ and it is easily shown that $\deg(f\alpha) = \deg(f)$. This continuous action of $G_\nu$ on $\mathfrak{F}$ gives $\mathfrak{F}$ the structure of a right $G_\nu$-module. Thus, the map $\deg : \mathfrak{F} \longrightarrow \mathbb{Z}$ is a continuous $G_\nu$-module homomorphism.

Defining the kernel of this map to be $\mathfrak{F}^0$ we once again have a short exact sequence of $G_\nu$-modules,

$$0 \longrightarrow \mathfrak{F}^0 \longrightarrow \mathfrak{F} \longrightarrow \mathbb{Z} \longrightarrow 0.$$

The corresponding long exact sequence of homology groups gives us a map,

$$\delta_{\mathfrak{F}} : H^0(G_\nu, \mathbb{Z}) = \mathbb{Z} \longrightarrow H^1(G_\nu, \mathfrak{F}^0),$$

and thus, once again, we have a canonical element $\delta_{\mathfrak{F}}(1) \in H^1(G_\nu, \mathfrak{F}^0)$.

Now, for a fundamental function $f \in \mathfrak{F}$ and for $\alpha \in G_\nu$, we define

$$\delta f(\alpha) := f\alpha - f.$$

Then, the function $\delta f$ is a continuous 1-cocycle representing $\delta_{\mathfrak{F}}(1)$.

**The Inner Product $< -|- >$**

For each $f,\ g \in \mathfrak{F}$ and $M \in \mathfrak{C}^0$ we define the inner product of $f$ and $g$ over $M$ to be,

$$< f, g|M > := \prod_{\xi \in \mu_m} \xi^{\int_{k_\nu^n} f(\mathcal{X}) g(\xi \mathcal{X}) M(\mathcal{X}) \mathrm{d}\mathcal{X}} \in \mu_m,$$

where this power of $\xi$ does indeed represent a well defined element of $\mu_m$.

Hill was able to show (in [3]) that this inner product satisfies the following:

- $< f, g|M >$ is trilinear in $f,\ g$ and $M$.
- The inner product is skew symmetric in $f$ and $g$ and satisfies

$$< f, f|M > = 1 \qquad \text{for any } f \in \mathfrak{F}.$$

- If $f$ and $g$ both have degree 0 then,

$$< f, g|M > = 1 \qquad \text{for every } M \in \mathfrak{C}^0.$$

- If $f,\ g$ and $h$ all have the same degree then,

$$< f, g|M >< g, h|M > = < f, h|M > \qquad \text{for every } M \in \mathfrak{C}^0.$$

- The inner product is $G_\nu$-covariant. That is,

$$< f\alpha, g\alpha|M\alpha > = < f, g|M >^\alpha \qquad \text{for every } \alpha \in G_\nu.$$

- The inner product is continuous on $\mathfrak{F} \times \mathfrak{F} \times \mathfrak{C}^0$ with respect to suitable topologies on $\mathfrak{F}$ and $\mathfrak{C}^0$.

Now, let us suppose that $g$ and $g'$ are two fundamental functions and $f \in \mathfrak{F}^0$. Then, using the properties described above, we have

$$\frac{< f, g|M >}{< f, g'|M >} = < f, g - g'|M > = 1, \qquad \text{for all } M \in \mathfrak{C}^0.$$

Thus, the inner product $< f, g|M >$ is independent of the fundamental function $g$.

**Definition:**

We define the $G_\nu$-module homomorphism,

$$< -|- >: \mathfrak{F}^0 \times \mathfrak{C}^0 \longrightarrow \mu_m, \qquad \text{by } < f|M > := < f, g|M >,$$

for each $f \in \mathfrak{F}^0$ and $M \in \mathfrak{C}^0$, and where $g$ is any fundamental function.

Since $< -|- >: \mathfrak{F}^0 \times \mathfrak{C}^0 \longrightarrow \mu_m$, is a bilinear map which commutes with the action of $G_\nu$ we have a cup product map,

$$\sqcup : H^1(G_\nu, \mathfrak{F}^0) \times H^1(G_\nu, \mathfrak{C}^0) \longrightarrow H^2(G_\nu, \mu_m).$$

Finally, by taking the cup product of the canonical elements $\delta_{\mathfrak{C}}(1) \in H^1(G_\nu, \mathfrak{C}^0)$ and $\delta_{\mathfrak{F}}(1) \in H^1(G_\nu, \mathfrak{F}^0)$ we are able to define a canonical element,

$$Dec := \delta_{\mathfrak{F}}(1) \sqcup \delta_{\mathfrak{C}}(1) \in H^2(G_\nu, \mu_m).$$

This class may be represented by a continuous 2-cocycle on $G_\nu$ with values in $\mu_m$ defined by,

$$\mathrm{Dec}_f^M(\alpha, \beta) := < f\alpha - f | M - M\beta^{-1} >^\beta,$$

where $f$ is any fundamental function and $M$ is any element of $\mathfrak{C}$ of degree 1.

### 1.4.2 Defining the cocycle $dec_\nu$

**Definition:**

Finally, we are able to define the cocycle $\mathrm{Dec}_f^M \in Z^2(\mathrm{GL}_n(k_\nu), \mu_m)$ corresponding to Hill's metaplectic cover of $\mathrm{GL}_n(k_\nu)$ by $\mu_m$,

$$1 \to \mu_m \longrightarrow \tilde{\mathcal{G}}_{dec_\nu} \longrightarrow \mathrm{GL}_n(k_\nu) \to 1.$$

Since the action of $G_\nu$ on $\mu_m$ is trivial, for each $\alpha$, $\beta \in G_\nu$, we define

$$\begin{aligned}\mathrm{Dec}_f^M(\alpha, \beta) &:= < f\alpha - f | M - M\beta^{-1} > \\ &= \prod_{\xi \in \mu_m} \xi^{\int_{k_\nu{}^n} f(\alpha\mathcal{X}) f(\xi\mathcal{X}) \left(M(\mathcal{X}) - M(\beta^{-1}\mathcal{X})\right) d\mathcal{X}},\end{aligned}$$

where $M : k_\nu{}^n \longrightarrow \mathbb{Z}$ is a continuous, $\mu_m$-invariant function such that $M(0) = 1$ and $f : k_\nu{}^n \backslash 0 \longrightarrow \mathbb{Z}$ is a continuous function satisfying,

$$\sum_{\xi \in \mu_m} f(\xi\mathcal{X}) = 1, \qquad \text{for all } \mathcal{X} \in k_\nu{}^n \backslash 0.$$

### 1.4.3 The functions $f$ and $M$

As we have seen, the cohomology class of the function $Dec$ does not depend on $f$ or $M$. However, the cocycle itself obviously does. Therefore, in order to calculate $\mathrm{Dec}_f^M$ we must first fix suitable functions $f$ and $M$.

#### The function $M$

Considering the properties satisfied by the function $M$, we see that when we restrict to one dimension and look at ${k_\nu}^\times = \mathrm{GL}_1(k_\nu)$ there appears to be a canonical choice for $M$.

If we define $M_1 : k_\nu \longrightarrow \mathbb{Z}$ to be the characteristic function of $\mathfrak{O}_\nu$, we have

$$M_1(x) = \begin{cases} 1 & x \in \mathfrak{O}_\nu, \\ 0 & x \in k_\nu \backslash \mathfrak{O}_\nu. \end{cases}$$

Then, as required, we find that $M_1$ is a continuous function which is $\mu_m$-invariant and satisfies $M_1(0) = 1$. So clearly, when defining the cocycle $\mathrm{Dec}_f^M$ on $\mathrm{GL}_1(k_\nu)$ we may take $M = M_1$.

#### The function $f$

To define a suitable function $f$ we shall once again restrict our attention to ${k_\nu}^\times = \mathrm{GL}_1(k_\nu)$.

Firstly we let $\mathcal{S} \subset (\mathfrak{O}_\nu/\wp_\nu) \setminus 0$ be an $m$-th set in $\mathfrak{O}_\nu/\wp_\nu$. That is, a complete set of representatives for $\mu_m$-orbits in $(\mathfrak{O}_\nu/\wp_\nu) \setminus 0$. We then define the characteristic function of $\mathcal{S}$ in $\mathfrak{O}_\nu/\wp_\nu$ to be,

$$f_{\mathcal{S}} : (\mathfrak{O}_\nu/\wp_\nu) \setminus 0 \longrightarrow \mathbb{Z}$$

such that, $$f_{\mathcal{S}}(a) = \begin{cases} 1 & a \in \mathcal{S} \\ 0 & a \notin \mathcal{S}. \end{cases}$$

Recalling that, in Section 1.2.1, we had chosen and fixed a uniformizer $\pi \in \mathfrak{O}_\nu$ we are now able to define the function $f_1 : k_\nu \backslash 0 \longrightarrow \mathbb{Z}$, by

$$f_1(\pi^X a) = f_{\mathcal{S}}(a \pmod{\pi}) \quad \text{for all } \forall \pi^X a \in k_\nu \backslash 0 \text{ with } \mid a \mid_\nu = 1.$$

Then, as required, $f_1$ is indeed a continuous function satisfying,

$$\sum_{\xi \in \mu_m} f_1(\xi \mathcal{X}) = 1, \quad \forall \mathcal{X} \in k_\nu \backslash 0.$$

Therefore, when we define the cocycle $\mathrm{Dec}_f^M$ on $k_\nu{}^\times = \mathrm{GL}_1(k_\nu)$ we may let $f = f_1$ be the function described above.

It should be noted that this function $f = f_1$ is clearly dependent on our earlier choice of the fixed uniformizer $\pi \in \mathfrak{O}_\nu$.

### Extending the functions $f_1$ and $M_1$ to $\mathrm{GL}_n$

Since the functions $f_1$ and $M_1$ for $k_\nu{}^\times = \mathrm{GL}_1(k_\nu)$ satisfy the conditions of our cocycle, we use them to build suitable functions $f_n$ and $M_n$ for any dimension $n$.

**Definition:**

Define $M_n : k_\nu{}^n \to \mathbb{Z}$ and $f_n : k_\nu{}^n \backslash 0 \to \mathbb{Z}$ by,

$$M_n(\mathcal{X}) = M_n((x_1, x_2, \dots, x_n)^T) = M_1(x_1)M_1(x_2)\dots M_1(x_n),$$

$$\begin{aligned} f_n(\mathcal{X}) &= f_n((x_1, x_2, \dots, x_n)^T) \\ &= f_1(x_i) \quad \iff |x_i|_\nu > |x_j|_\nu, \forall j < i, \quad \text{and} \quad |x_i|_\nu \geq |x_k|_\nu, \forall k > i, \end{aligned}$$

then these functions, $f_n$ and $M_n$, clearly satisfy the conditions given in the definition of our cocycle.

**Lemma 1.4.1** *The function $f_n$ satisfies the additional property,*

$$f_n\pi = f_n.$$

**PROOF:**
The proof of this statement follows directly from the fact that $f_1\pi = f_1$. □

**Definition:**

We are now able to define the cocycle $dec_\nu \in Z^2(\mathrm{GL}_n(k_\nu), \mu_m)$ with which this thesis is concerned. That is, we define

$$dec_\nu(\alpha, \beta) := \mathrm{Dec}_f^M(\alpha, \beta),$$

where the functions $f := f_n$ and $M := M_n$ are precisely the functions described above.

**note:** From now on we shall use the notation $f := f_n$ and $M := M_n$, using the subscripts only when the dimension is to be emphasized.

**Remark:**

Considering the definition, we see that the function $f := f_n$ will clearly depend on our earlier choice of uniformizer $\pi \in \mathfrak{O}_\nu$. However, this means that the cocycle $dec_\nu$ will also depend on the choice of $\pi$. Therefore, throughout this thesis, as in Section 1.2.1, we insist that the uniformizer $\pi$ has been chosen and fixed.

**Proposition 1.4.1** *The cocycle $dec_\nu := \mathrm{Dec}_f^M \in Z^2(\mathrm{GL}_n(k_\nu), \mu_m)$ is independent of the choice of m-th set $\mathcal{S}$ used in the definition of the fundamental function $f$.*

**PROOF:**

Let us begin by considering the cocycle $dec_\nu$ over the group $\mu_q = (\mathfrak{O}_\nu/\wp_\nu)^\times$ of roots of unity. As we have seen, on $\mathrm{GL}_1$, the function $f := f_1$ satisfies

$$f_1(\pi^X a) = f_{\mathcal{S}}(a \pmod{\pi}),$$

where $\mathcal{S} \subset (\mathfrak{O}_\nu/\wp_\nu) \setminus 0$ is a $q$-th set in $\mathfrak{O}_\nu/\wp_\nu$.

Let us now consider the cocycle $\mathrm{Dec}_g^M$ over $\mu_q$, where the fundamental function $g : {k_\nu}^n \setminus 0 \longrightarrow \mathbb{Z}$ is defined as follows:

$$g(\mathcal{X}) := g_n(\mathcal{X}) = g_1(x_i) \iff |x_i|_\nu > |x_j|_\nu, \forall j < i, \quad \text{and} \quad |x_i|_\nu \geq |x_k|_\nu, \forall k > i,$$

such that,

$$g_1(\pi^X a) = g_{\mathcal{T}}(a \pmod{\pi}) \qquad \text{where,} \quad g_{\mathcal{T}}(\mathrm{a}) = \begin{cases} 1 & a \in \mathcal{T} \\ 0 & a \notin \mathcal{T}, \end{cases}$$

and where $\mathcal{T} \neq \mathcal{S}$ is a different $q$-th set in $\mathfrak{O}_\nu/\wp_\nu$.

Since we are considering both the cocycles $\mathrm{Dec}_f^M$ and $\mathrm{Dec}_g^M$ to have values in $\mu_q = (\mathfrak{O}_\nu/\wp_\nu)^\times$ and both $\mathcal{T}$ and $\mathcal{S}$ are $q$-th sets in $\mathfrak{O}_\nu/\wp_\nu$, we must have $\mathcal{S} = \varepsilon\, \mathcal{T}$ for some $\varepsilon \in \mu_q$. Using this fact and considering the definitions of the functions $f := f_n$ and $g := g_n$ we may immediately deduce that,

$$\begin{aligned} g_{\mathcal{T}}(a) = 1 \qquad &\Leftrightarrow \qquad f_{\mathcal{S}}(\varepsilon a) = 1 \\ \Rightarrow \qquad g_1 = f_1 \varepsilon \qquad &\Rightarrow \qquad g_n = f_n \varepsilon. \end{aligned}$$

Let us now consider the quotient of the two cocycles $\mathrm{Dec}_f^M$, $\mathrm{Dec}_g^M \in Z^2(\mathrm{GL}_n(k_\nu), \mu_q)$. That is,

$$\frac{\mathrm{Dec}_f^M(\alpha,\beta)}{\mathrm{Dec}_g^M(\alpha,\beta)} = \frac{< f\alpha - f|M - M\beta^{-1} >}{< g\alpha - g|M - M\beta^{-1} >}$$

$$= < (f-g)\alpha - (f-g)|M - M\beta^{-1} >$$

$$= \frac{< (f-g)|(M - M\beta^{-1})\alpha^{-1} >}{< (f-g)|M - M\beta^{-1} >}$$

$$= < (f-g)|M\alpha^{-1} + M\beta^{-1} - M - M\alpha^{-1}\beta^{-1} >$$

$$= < (f-f\varepsilon)|M\alpha^{-1} + M\beta^{-1} - M - M\alpha^{-1}\beta^{-1} >$$

$$= \prod_{\xi\in\mu_q} \xi^{\int_{k_\nu{}^n} f(\varepsilon\mathcal{X})f(\xi\mathcal{X})\left(M(\alpha^{-1}\mathcal{X})+M(\beta^{-1}\mathcal{X})-M(\mathcal{X})-M(\alpha^{-1}\beta^{-1}\mathcal{X})\right)\mathrm{d}\mathcal{X}}.$$

However, this exponent will be zero unless $\xi = \epsilon$. Noting that $f(\epsilon\mathcal{X}) = f(\epsilon\mathcal{X})^2$, our expression becomes,

$$\frac{\mathrm{Dec}_f^M(\alpha,\beta)}{\mathrm{Dec}_g^M(\alpha,\beta)} = \varepsilon^{\int_{k_\nu{}^n} f(\varepsilon\mathcal{X})\left(M(\alpha^{-1}\mathcal{X})+M(\beta^{-1}\mathcal{X})-M(\mathcal{X})-M(\alpha^{-1}\beta^{-1}\mathcal{X})\right)\mathrm{d}\mathcal{X}}.$$

Finally, by using the result of Lemma 17 in Hill's paper [3], we have

$$= \varepsilon^{\frac{1}{q}\int_{k_\nu{}^n} \left(M(\alpha^{-1}\mathcal{X})+M(\beta^{-1}\mathcal{X})-M(\mathcal{X})-M(\alpha^{-1}\beta^{-1}\mathcal{X})\right)\mathrm{d}\mathcal{X}}$$

$$= \varepsilon^{\frac{1}{q}(|\det(\alpha)|_\nu + |\det(\beta)|_\nu - 1 - |\det(\alpha\beta)|_\nu)}$$

$$= \varepsilon^{\frac{1}{q}(|\det(\alpha)|_\nu - 1)(|\det(\beta)|_\nu - 1)} = 1,$$

since $(|\det(\alpha)|_\nu - 1)(|\det(\beta)|_\nu - 1) \equiv 0 \pmod{q^2}$ and $\varepsilon \in \mu_q$ is a $q$-th root of unity.

Using the transfer map (Thm 6 [3]) it may be shown that the cocycle $dec_\nu$, when restricted of to some group of roots $\mu_m \subset \mu_q$, is simply some power of the cocycle on $\mu_q$. Therefore we are finally able to conclude that,

$$dec_\nu := \mathrm{Dec}_g^M = \mathrm{Dec}_f^M,$$

where $dec_\nu \in Z^2(\mathrm{GL}_n(k_\nu), \mu_m)$ has values in any group of roots $\mu_m \subset \mu_q$.

That is, the cocycle $dec_\nu$ is indeed independent of our choice of $m$-th set $\mathcal{S}$. □

This result implies that our choice of $f$ was arbitrary with respect to the fact that any function chosen in this way gives us an identical cocycle. Therefore, as with the function $M$, the choice of the function $f$ appears also to be canonical.

In order to conclude this chapter we shall consider a couple of results for the cocycle $dec_\nu$ which are found as a direct consequence of our choice of functions $f$ and $M$. These results shall be used repeatedly throughout the proceeding chapters.

**Theorem 1.4.1** *Let $g_1$, $g_2 \in GL_n(k_\nu)$, then the cocycle $dec_\nu$ satisfies,*

$$dec_\nu(g_1, g_2) \;=\; 1,$$

*whenever the matrix $g_2 \in GL_n(\mathfrak{O}_\nu)$.*

**PROOF OF THEOREM:**

Using the definition of the cocycle $dec_\nu$ we have,

$$dec_\nu(g_1, g_2) \;= < fg_1 - f|M - Mg_2^{-1} > .$$

Now, since $M$ is the characteristic function of $\mathfrak{O}_\nu^n$, whenever $g_2 \in \mathrm{GL}_n(\mathfrak{O}_\nu)$, we are able to deduce that,

$$\begin{aligned} M(\mathcal{X}) = 1 \iff M(g_2^{-1}\mathcal{X}) = 1 \qquad & \text{for each } \mathcal{X} \in {k_\nu}^n, \\ \Rightarrow \qquad Mg_2^{-1} = M, & \end{aligned}$$

from which our result quickly follows. □

In order to prove the final theorem in this chapter we shall first require a few lemmas concerning the function $f$.

**Lemma 1.4.2** *Let $v = (v_1, \ldots, v_n)^T \in V$ where we define the set $V = \mathfrak{O}_\nu^n \setminus \pi\mathfrak{O}_\nu^n$.*
*Then $f(v)$ depends only on the value of $v \pmod{\pi\mathfrak{O}_\nu^n}$.*

**PROOF:**

Since we know that $|v|_\nu := \max\{|v_i|_\nu : 1 \le i \le n\} = 1$ we know that there exists an integer $j$ with $1 \le j \le n$ such that,

$$v_j \not\equiv 0 \pmod{\pi} \quad \text{but,} \quad v_k \equiv 0 \pmod{\pi} \quad \text{for all } k < j.$$

Therefore we must have,

$$f(v) = f(v_j) = f_S(v_j \pmod{\pi}),$$

which proves our original statement. □

**Lemma 1.4.3** *Let the matrix $g_1 \equiv I_n \pmod{\pi}$ and also let $v \in V$, where $V$ is as previously defined. Then,*

$$f(g_1 v) \;=\; f(v).$$

**PROOF:**

The proof of this statement simply requires us to notice that,

$$\begin{aligned} g_1 v - v = (g_1 - I_n)v \in \pi\mathfrak{O}_\nu^n \\ \Rightarrow \quad g_1 v \equiv v \pmod{\pi\mathfrak{O}_\nu^n}. \end{aligned}$$

Then, our result is a simple consequence of the previous lemma. □

**Lemma 1.4.4** *Let the matrix* $g_1 \equiv I_n \pmod{\pi}$, *then the function* $f$ *satisfies,*

$$fg_1 = f.$$

**PROOF:**

If we let $v \in {k_\nu}^n \backslash 0$ then there exists a unique integer $\Lambda \in \mathbb{Z}$ such that,

$$v_0 = \pi^\Lambda v \in \mathfrak{O}_\nu^n \setminus \pi\mathfrak{O}_\nu^n = V.$$

However, this enables us to use the previous two Lemmas together with Lemma 1.4.1 to conclude that,

$$\begin{aligned} f(v) \;=\; f(v_0) \;=\; f(g_1 v_0) \;=\; f(g_1 v), \\ \Rightarrow \quad fg_1 = f. \end{aligned}$$

□

Finally, we conclude this chapter with the following important theorem concerning the cocycle $dec_\nu$.

**Theorem 1.4.2** *Let* $g_1,\ g_2 \in GL_n(k_\nu)$, *then the cocycle* $dec_\nu$ *satisfies*

$$dec_\nu(g_1, g_2) \;=\; 1,$$

*whenever the matrix* $g_1 \equiv I_n \pmod{\pi}$.

**PROOF OF THEOREM:**

Since the cocycle $dec_\nu$ is defined to be,

$$dec_\nu(g_1, g_2) \;= < fg_1 - f | M - Mg_2^{-1} >,$$

and the previous lemma states that $fg_1 = f$ we do indeed find that,

$$dec_\nu(g_1, g_2) \;=\; 1.$$

□

# Chapter 2

# Calculation of $dec_\nu$ on the Torus

We begin by letting $k_\nu$ be a local field with valuation $\nu$ and a fixed uniformizing element $\pi$ in $\mathfrak{O}_\nu$ the ring of integers in $k_\nu$. In this chapter we shall restrict our attention to the torus $T$ of diagonal matrices in $\mathrm{GL}_n(k_\nu)$ and calculate explicit formulae for the 2-cocycle $dec_\nu \in Z^2(\mathrm{GL}_n(k_\nu), \mu_m)$.

Having fixed $k_\nu$ and $\pi$ we know that for each $\alpha_n, \beta_n \in T \subset \mathrm{GL}_n(k_\nu)$ the cocycle satisfies,

$$\begin{aligned} dec_\nu(\alpha_n, \beta_n) &:= < f\alpha_n - f | M - M\beta_n^{-1} > \\ &= \prod_{\xi \in \mu_m} \xi^{\int_{k_\nu{}^n} f(\alpha_n \mathcal{X}) f(\xi \mathcal{X}) \left(M(\mathcal{X}) - M(\beta_n^{-1}\mathcal{X})\right) \mathrm{d}\mathcal{X}}, \end{aligned}$$

where the functions $f$ and $M$ are as previously defined and where the integral is taken over $k_\nu{}^n$ with respect to a Haar measure normalized such that,

$$\int_{k_\nu{}^n} M(\mathcal{X}) \mathrm{d}\mathcal{X} = 1.$$

We shall calculate this expression by first considering the cases when $n = 1,\ 2$ and 3. We shall then proceed by induction to include the general case for any dimension $n$.

In order to begin this inductive process and calculate the integral in the case that $n = 1$ we shall need to employ the Gauss-Schering Lemma. For the purposes of this chapter we shall state this as:

**Lemma 2.0.5**

$$\prod_{\xi \in \mu_m} \xi^{\sum_{\mathcal{X} \in (\mathfrak{O}_\nu \setminus \pi \mathfrak{O}_\nu) \setminus 0} f(a\mathcal{X}) f(\xi \mathcal{X})} = (a, \pi)_{\nu,m},$$

*where $( - , - )_{\nu,m}$ is the local Hilbert symbol as previously described in the introduction.*

**PROOF:**

The proof of this statement can be found in Hill's paper [4]. □

## Notation

Let us remind ourselves that, having fixed a uniformizer $\pi$ in the ring of integers $\mathfrak{O}_\nu \subset k_\nu$, the diagonal matrices $\alpha_n$, $\beta_n \in T$ have previously been defined as,

$$\alpha_n = \mathrm{diag}(\pi^{X_1}a_1, \dots, \pi^{X_n}a_n), \qquad \beta_n = \mathrm{diag}(\pi^{Y_1}b_1, \dots, \pi^{Y_n}b_n).$$

Let us also be clear that we define the variables $\mathcal{X} \in {k_\nu}^n$ and $\hat{\mathcal{X}} \in {k_\nu}^{n-1}$ such that,

$$\mathcal{X} = (x_1, \dots, x_n)^T \quad \text{and} \quad \hat{\mathcal{X}} = (x_1, \dots, x_{n-1})^T.$$

We shall also use the notation,

$$\begin{aligned} |\mathcal{X}|_\nu = \quad & |(x_1, \dots, x_n)^T|_\nu &= \max\{|x_1|_\nu, \dots, |x_n|_\nu\} \\ |\hat{\mathcal{X}}|_\nu = \quad & |(x_1, \dots, x_{n-1})^T|_\nu &= \max\{|x_1|_\nu, \dots, |x_{n-1}|_\nu\}. \end{aligned}$$

Before we begin the calculations we shall first refer to page 756 of Hill's paper [3], in which he proves the following important lemma:

**Lemma 1** *For all $M \in \mathfrak{C}$ one has*

$$\int_{{k_\nu}^n} M(\mathcal{X})\mathrm{d}\mathcal{X} \equiv \deg(M) \pmod{m}.$$

note: In Lemma 16, Hill proves slightly more than this. Instead of integrating over ${k_\nu}^n$, we may instead integrate over an abelian, totally disconnected, Hausdorff, locally compact topological group.

This lemma is important because it allows us to deduce the following:

If M is the characteristic function of $\Lambda$ a compact, open, $\mu_m$-invariant neighbourhood of zero in ${k_\nu}^n$, then

$$\int_\Lambda M(\mathcal{X})\mathrm{d}\mathcal{X} \equiv M(0) = 1 \pmod{m}.$$

## 2.1 The cocycle $dec_\nu$ on ${k_\nu}^\times = \mathrm{GL}_1(k_\nu)$

**Theorem 2.1.1** *Let $k_\nu$ be a local field with valuation $\nu$. Then, for each $\alpha_1 = \pi^X a$, $\beta_1 = \pi^Y b \in {k_\nu}^\times$ the cocycle $dec_\nu$ satisfies,*

$$dec_\nu(\pi^X a, \pi^Y b) = (a, \pi)^Y_{\nu,m}. \tag{2.1}$$

In order to prove this we shall first consider a few simple lemmas:

**Lemma 2.1.1** *For each $a \in k_\nu{}^\times$ the cocycle $dec_\nu$ satisfies,*

$$dec_\nu(a,\pi) = (a,\pi)_{\nu,m}.$$

**PROOF:** By definition we have,

$$dec_\nu(a,\pi) = \prod_{\xi\in\mu_m} \xi^{\{\int_{\mathcal{O}_\nu} - \int_{\pi\mathcal{O}_\nu}\} f(a\mathcal{X})f(\xi\mathcal{X})d\mathcal{X}}$$

changing the integral for a sum this expression becomes,

$$= \prod_{\xi\in\mu_m} \xi^{\sum_{\mathcal{X}\in(\mathcal{O}_\nu\setminus\pi\mathcal{O}_\nu)\setminus 0} f(a\mathcal{X})f(\xi\mathcal{X})}$$

which, by the Gauss-Schering Lemma, is found to be

$$= (a,\pi)_{\nu,m}.$$

□

**Lemma 2.1.2** *For every $\pi^X a$, $\pi^Y b \in k_\nu{}^\times$ the cocycle $dec_\nu$ satisfies,*

$$dec_\nu(\pi^X a, \pi^Y b) = dec_\nu(a, \pi^Y).$$

**PROOF:** Using the notation of section 1.4.2 from the previous chapter, we may write this as,

$$dec_\nu(\pi^X a, \pi^Y b) = < f\pi^X a - f | M - M\pi^{-Y} b^{-1} > .$$

However, using Lemma 1.4.1 and Theorem 1.4.1 from section 1.4.3, we find that the functions $f$ and $M$ satisfy,

$$f\pi = f \qquad \text{and} \qquad Mb^{-1} = M.$$

Therefore we may conclude that,

$$\begin{aligned} dec_\nu(\pi^X a, \pi^Y b) &= < f\pi^X a - f | M - M\pi^{-Y} b^{-1} > \\ &= < fa - f | M - M\pi^{-Y} > = dec_\nu(a, \pi^Y). \end{aligned}$$

□

**PROOF OF THEOREM:** In order to prove Theorem 2.1.1 we need simply use the previous two Lemmas and apply the cocycle rule (see page 14) inductively. That is,

$$\begin{aligned} dec_\nu(\pi^X a, \pi^Y b) &= dec_\nu(a, \pi^Y) \\ &= dec_\nu(a, \pi.\pi^{Y-1}) \\ &= dec_\nu(a,\pi).\, dec_\nu(a, \pi^{Y-1}) = \ldots \\ &= dec_\nu(a,\pi)^Y \\ &= (a,\pi)^Y_{\nu,m}. \end{aligned}$$

□

## 2.2 The cocycle $dec_\nu$ on $T \subset \mathrm{GL}_2(k_\nu)$

**Theorem 2.2.1** *Let $k_\nu$ be a local field with valuation $\nu$. Then there exists an explicit function,*

$$\tau_2 : T \longrightarrow \{1, -1\},$$

*such that for each $\alpha_2$, $\beta_2 \in T \subset GL_2(k_\nu)$ the cocycle $dec_\nu$ satisfies,*

$$\begin{aligned} dec_\nu(\alpha_2, \beta_2) &= dec_\nu(\pi^{X_1}a_1, \pi^{Y_1}b_1)\, dec_\nu(\pi^{X_2}a_2, \pi^{Y_2}b_2)\frac{\tau_2(\alpha_2)\tau_2(\beta_2)}{\tau_2(\alpha_2\beta_2)}(-1)^{\frac{(\rho-1)}{m}X_2Y_1} \\ &= (a_1^{Y_1}a_2^{Y_2}, \pi)_{\nu,m}\frac{\tau_2(\alpha_2)\tau_2(\beta_2)}{\tau_2(\alpha_2\beta_2)}(-1)^{\frac{(\rho-1)}{m}X_2Y_1}, \end{aligned} \tag{2.2}$$

*where $\tau_2$ and, as we have seen previously, $(-1)$ are both trivial if $m$ is odd.*

The function $\tau_2$ will be described in more detail in chapter 3.

**PROOF OF THEOREM:** Referring to the definition of the cocycle we find,

$$dec_\nu(\alpha_2, \beta_2) = \prod_{\xi \in \mu_m} \xi^{\int_{k_\nu{}^2\backslash 0} f(\alpha_2\mathcal{X})f(\xi\mathcal{X})\left(M(\mathcal{X})-M(\beta_2^{-1}\mathcal{X})\right)\mathrm{d}\mathcal{X}} =: \prod_{\xi\in\mu_m} \xi^{I(\xi)},$$

where we define the integral,

$$I(\xi) = \int_{k_\nu{}^2\backslash 0} f(\alpha_2\mathcal{X})f(\xi\mathcal{X})\big(M(\mathcal{X}) - M(\beta_2^{-1}\mathcal{X})\big)\mathrm{d}\mathcal{X}.$$

In order to calculate this integral $I(\xi)$ we first decompose $k_\nu{}^2\backslash 0$ into four disjoint open sets,

$$\begin{aligned} A_1 &= \{(x_1, x_2)^T \in k_\nu{}^2\backslash 0 : |x_1|_\nu \geq |x_2|_\nu, |\pi^{X_1}a_1x_1|_\nu \geq |\pi^{X_2}a_2x_2|_\nu\} \\ A_2 &= \{(x_1, x_2)^T \in k_\nu{}^2\backslash 0 : |x_1|_\nu \geq |x_2|_\nu, |\pi^{X_1}a_1x_1|_\nu < |\pi^{X_2}a_2x_2|_\nu\} \\ A_3 &= \{(x_1, x_2)^T \in k_\nu{}^2\backslash 0 : |x_1|_\nu < |x_2|_\nu, |\pi^{X_1}a_1x_1|_\nu \geq |\pi^{X_2}a_2x_2|_\nu\} \\ A_4 &= \{(x_1, x_2)^T \in k_\nu{}^2\backslash 0 : |x_1|_\nu < |x_2|_\nu, |\pi^{X_1}a_1x_1|_\nu < |\pi^{X_2}a_2x_2|_\nu\}. \end{aligned}$$

Enabling us to write,

$$dec_\nu(\alpha_2, \beta_2) = \prod_{\xi\in\mu_m} \xi^{\sum_{i=1}^4 I_i(\xi)}.$$

where, $$I_i(\xi) = \int_{A_i} f(\alpha_2\mathcal{X})f(\xi\mathcal{X})\big(M(\mathcal{X}) - M(\beta_2^{-1}\mathcal{X})\big)\mathrm{d}\mathcal{X}.$$

Before we begin evaluating the integrals let us consider the value of our function $f$ on each of our sets $A_i$.

$$\begin{array}{llll} \mathcal{X} \in A_1 & \Rightarrow & f(\alpha_2\mathcal{X}) = f(\pi^{X_1}a_1x_1), & f(\xi\mathcal{X}) = f(\xi x_1) \\ \mathcal{X} \in A_2 & \Rightarrow & f(\alpha_2\mathcal{X}) = f(\pi^{X_2}a_2x_2), & f(\xi\mathcal{X}) = f(\xi x_1) \\ \mathcal{X} \in A_3 & \Rightarrow & f(\alpha_2\mathcal{X}) = f(\pi^{X_1}a_1x_1), & f(\xi\mathcal{X}) = f(\xi x_2) \\ \mathcal{X} \in A_4 & \Rightarrow & f(\alpha_2\mathcal{X}) = f(\pi^{X_2}a_2x_2), & f(\xi\mathcal{X}) = f(\xi x_2). \end{array}$$

For reasons which shall become clear it is convenient for us to look at these four integrals in pairs and calculate them together.

**The integrals $I_1(\xi)$ and $I_4(\xi)$:**

Considering the table of results given above, the integrals $I_1(\xi)$ and $I_4(\xi)$ satisfy,

$$I_1(\xi) = \int_{A_1} f(\pi^{X_1} a_1 x_1) f(\xi x_1)\big(M(\mathcal{X}) - M(\beta_2^{-1}\mathcal{X})\big)\mathrm{d}\mathcal{X}$$
$$I_4(\xi) = \int_{A_4} f(\pi^{X_2} a_2 x_2) f(\xi x_2)\big(M(\mathcal{X}) - M(\beta_2^{-1}\mathcal{X})\big)\mathrm{d}\mathcal{X}.$$

For all $x_1,\ x_2 \in k_\nu$, let us define the sets $A_1(x_1)$ and $A_4(x_2)$ by,

$$A_1(x_1) = \{x_2 \in k_\nu : (x_1, x_2)^T \in A_1\}, \qquad A_4(x_2) = \{x_1 \in k_\nu : (x_1, x_2)^T \in A_4\}.$$

Then both sets are compact, open, $\mu_m$-invariant neighbourhoods of zero in $k_\nu$.

By restriction, $M : A_1(x_1) \to \mathbb{Z}$ is the characteristic function of $A_1(x_1) \cap \mathfrak{O}_\nu$ and similarly $M : A_4(x_2) \to \mathbb{Z}$ is the characteristic function of $A_4(x_2) \cap \mathfrak{O}_\nu$. Since both of these spaces are compact, open, $\mu_m$-invariant neighbourhoods of zero in $k_\nu$ we have satisfied the conditions of Lemma 1. Therefore we find,

$$\begin{aligned}
I_1(\xi) &= \int_{k_\nu} f(\pi^{X_1} a_1 x_1) f(\xi x_1) \int_{A_1(x_1)} \big(M(\mathcal{X}) - M(\beta_2^{-1}\mathcal{X})\big)\mathrm{d}x_1\mathrm{d}x_2 \\
&= \int_{k_\nu} f(\pi^{X_1} a_1 x_1) f(\xi x_1) \left\{ \int_{A_1(x_1)} M(x_1)M(x_2) - M(\pi^{-Y_1} b_1^{-1} x_1) M(\pi^{-Y_2} b_2^{-1} x_2)\mathrm{d}x_2 \right\}\mathrm{d}x_1 \\
&= \int_{k_\nu} f(\pi^{X_1} a_1 x_1) f(\xi x_1) \left\{ \left(M(x_1)\int_{A_1(x_1)} M(x_2)\mathrm{d}x_2\right) - \left(M(\pi^{-Y_1} b_1^{-1} x_1)\int_{A_1(x_1)} M(\pi^{-Y_2} b_2^{-1} x_2)\mathrm{d}x_2\right) \right\}\mathrm{d}x_1 \\
&\equiv \int_{k_\nu} f(\pi^{X_1} a_1 x_1) f(\xi x_1)\big(M(x_1) - M(\pi^{-Y_1} b_1^{-1} x_1)\big)\mathrm{d}x_1 \qquad (\mathrm{mod}\ m).
\end{aligned}$$

Similarly we also find that,

$$\begin{aligned}
I_4(\xi) &= \int_{k_\nu} f(\pi^{X_2} a_2 x_2) f(\xi x_2) \left\{ \left(M(x_2)\int_{A_4(x_2)} M(x_1)\mathrm{d}x_1\right) - \left(M(\pi^{-Y_2} b_2^{-1} x_2)\int_{A_4(x_2)} M(\pi^{-Y_1} b_1^{-1} x_1)\mathrm{d}x_1\right) \right\}\mathrm{d}x_2 \\
&\equiv \int_{k_\nu} f(\pi^{X_2} a_2 x_2) f(\xi x_2)\big(M(x_2) - M(\pi^{-Y_2} b_2^{-1} x_2)\big)\mathrm{d}x_2 \qquad (\mathrm{mod}\ m).
\end{aligned}$$

Putting our results back together, we may conclude that,

$$\prod_{\xi\in\mu_m} \xi^{I_1(\xi)+I_4(\xi)} = dec_\nu(\pi^{X_1}a_1, \pi^{Y_1}b_1)\, dec_\nu(\pi^{X_2}a_2, \pi^{Y_2}b_2).$$

**The integrals $I_2(\xi)$ and $I_3(\xi)$:**

We must now calculate the integrals,

$$I_2(\xi) = \int_{A_2} f(\pi^{X_2}a_2x_2)f(\xi x_1)\big(M(\mathcal{X}) - M(\beta_2^{-1}\mathcal{X})\big)\mathrm{d}\mathcal{X}$$
$$I_3(\xi) = \int_{A_3} f(\pi^{X_1}a_1x_1)f(\xi x_2)\big(M(\mathcal{X}) - M(\beta_2^{-1}\mathcal{X})\big)\mathrm{d}\mathcal{X}.$$

In order to solve these integrals we must consider the action of $\mu_m \oplus \mu_m$ on $k_\nu \oplus k_\nu$. Under this action it is clear that,

$$A_2,\ A_3 \quad \text{and} \quad \big(M(\mathcal{X}) - M(\beta_2^{-1}\mathcal{X})\big)\mathrm{d}\mathcal{X}$$

are $\mu_m \oplus \mu_m$-invariant.

Thus, using this action of $\mu_m \oplus \mu_m$, for the integral $I_2(\xi)$ we calculate,

$$\begin{aligned} m^2 I_2(\xi) &= \sum_{\xi_1,\xi_2\in\mu_m} \int_{A_2} f(\xi_1\pi^{X_2}a_2x_2)f(\xi_2\xi x_1)\big(M(\mathcal{X}) - M(\beta_2^{-1}\mathcal{X})\big)\mathrm{d}\mathcal{X} \\ &= \int_{A_2} \Big(\sum_{\xi_1,\xi_2\in\mu_m} f(\xi_1\pi^{X_2}a_2x_2)f(\xi_2\xi x_1)\Big)\big(M(\mathcal{X}) - M(\beta_2^{-1}\mathcal{X})\big)\mathrm{d}\mathcal{X}. \end{aligned}$$

Now, using the fact that $f$ is a fundamental function, this simply becomes

$$m^2 I_2(\xi) = \int_{A_2} \big(M(\mathcal{X}) - M(\beta_2^{-1}\mathcal{X})\big)\mathrm{d}\mathcal{X}.$$

Similarly, for $I_3(\xi)$, we find

$$m^2 I_3(\xi) = \int_{A_3} \big(M(\mathcal{X}) - M(\beta_2^{-1}\mathcal{X})\big)\mathrm{d}\mathcal{X}.$$

In particular we see that both $I_2(\xi) := I_2$ and $I_3(\xi) := I_3$ are independent of $\xi$. Therefore we are able to write,

$$\prod_{\xi\in\mu_m} \xi^{I_2(\xi)+I_3(\xi)} = \Big(\prod_{\xi\in\mu_m} \xi\Big)^{I_2(\xi)+I_3(\xi)} = (-1)^{I_2+I_3},$$

where this expression is trivial when $m$ is odd.

Since we are now only interested in $I_2 + I_3 \pmod 2$, we actually calculate $I_2 - I_3$. In order to do this we first define the set $S_2$ by,

$$S_2 = \{(x_1, x_2)^T \in k_\nu \oplus k_\nu : |x_1|_\nu \geq |x_2|_\nu\}.$$

Then, using this definition we are able to write,

$$A_2 - A_3 = (A_2 + A_1) - (A_3 + A_1) = S_2 - \alpha_2^{-1} S_2.$$

Thus the difference between the integrals $I_2$ and $I_3$ may be expressed as,

$$\begin{aligned} I_2 - I_3 &= \frac{1}{m^2}\left\{\int_{S_2} - \int_{\alpha_2^{-1}S_2}\right\}(M(\mathcal{X}) - M(\beta_2^{-1}\mathcal{X}))\mathrm{d}\mathcal{X} \\ &= \frac{1}{m^2}\int_{S_2}(M(\mathcal{X}) - M(\beta_2^{-1}\mathcal{X}))\mathrm{d}\mathcal{X} \\ &\qquad - |\det(\alpha_2)|_\nu^{-1}\int_{S_2}(M(\alpha_2^{-1}\mathcal{X}) - M((\alpha_2\beta_2)^{-1}\mathcal{X}))\mathrm{d}\mathcal{X}. \end{aligned}$$

By defining the function $J_2$ on the torus in $\mathrm{GL}_2(k_\nu)$ by,

$$J_2(\alpha_2) = \int_{S_2}(M(\mathcal{X}) - M(\alpha_2^{-1}\mathcal{X}))\mathrm{d}\mathcal{X},$$

and substituting into the expression above we are able to write,

$$I_2 - I_3 = \frac{1}{m^2}\Big(J_2(\beta_2) + |\det(\alpha_2)|_\nu^{-1} J_2(\alpha_2) - |\det(\alpha_2)|_\nu^{-1} J_2(\alpha_2\beta_2)\Big).$$

Furthermore, let us define the function $\jmath_2$ by,

$$\jmath_2(\alpha_2) := (-1)^{\frac{1}{m^2}\left(J_2(\alpha_2) - (1 - |\pi^{X_1}a_1|_\nu)\right)},$$

where, since it has been shown by Hill (Page 761, [3]) that the function $J_2$ satisfies,

$$J_2(\alpha_2) \equiv 1 - |\pi^{X_1}a_1|_\nu \pmod{m^2}.$$

this definition does indeed make sense.

Using this new cochain we are finally able to calculate,

$$\begin{aligned} (-1)^{I_2+I_3} &= \jmath_2(\beta_2).\left(\frac{\jmath_2(\alpha_2)}{\jmath_2(\alpha_2\beta_2)}\right)^{|\det(\alpha_2)|_\nu^{-1}}(-1)^{\frac{1}{m^2}\left(1 - \frac{1}{\rho^{Y_1}} + \frac{1}{\rho^{X_1+X_2}}\left(\frac{1}{\rho^{X_1+Y_1}} - \frac{1}{\rho^{X_1}}\right)\right)} \\ &= \frac{\jmath_2(\alpha_2)\jmath_2(\beta_2)}{\jmath_2(\alpha_2\beta_2)}(-1)^{\frac{1}{m^2}\left(1-\frac{1}{\rho^{Y_1}}\right)\left(1 - \frac{1}{\rho^{2X_1}}\cdot\frac{1}{\rho^{X_2}}\right)} \\ &= \frac{\jmath_2(\alpha_2)\jmath_2(\beta_2)}{\jmath_2(\alpha_2\beta_2)}(-1)^{\frac{1}{m^2}\left(1-\frac{1}{\rho^{Y_1}}\right)\left(1 - \frac{1}{\rho^{X_2}}\right)} \\ &= \frac{\jmath_2(\alpha_2)\jmath_2(\beta_2)}{\jmath_2(\alpha_2\beta_2)}(-1)^{\frac{(\rho-1)}{m}Y_1X_2}. \end{aligned}$$

In conclusion, on the torus $T \subset \mathrm{GL}_2(k_\nu)$, we are able to deduce that the cocycle $dec_\nu$ satisfies,

$$\begin{aligned} dec_\nu(\alpha_2, \beta_2) &= dec_\nu(\pi^{X_1}a_1, \pi^{Y_1}b_1)\, dec_\nu(\pi^{X_2}a_2, \pi^{Y_2}b_2)\frac{\jmath_2(\alpha_2)\jmath_2(\beta_2)}{\jmath_2(\alpha_2\beta_2)}(-1)^{\frac{(\rho-1)}{m}Y_1X_2} \\ &= (a_1^{Y_1}a_2^{Y_2}, \pi)_{\nu,m}\frac{\tau_2(\alpha_2)\tau_2(\beta_2)}{\tau_2(\alpha_2\beta_2)}(-1)^{\frac{(\rho-1)}{m}Y_1X_2}, \end{aligned} \tag{2.3}$$

where the function $\tau_2 = \jmath_2$, as described above. The reason for re-naming this function will become clear when we come to calculate it in the next chapter.

□

## 2.3 The cocycle $dec_\nu$ on $T \subset \mathrm{GL}_3(k_\nu)$

**Theorem 2.3.1** *Let $k_\nu$ be a local field with valuation $\nu$. Then there exists an explicit function,*

$$\tau_3 : T \longrightarrow \{1, -1\},$$

*such that, for each $\alpha_3$, $\beta_3 \in T \subset GL_3(k_\nu)$ the cocycle $dec_\nu$ satisfies,*

$$\begin{aligned} dec_\nu(\alpha_3, \beta_3) &= \prod_{i-1}^{3} dec_\nu(\pi^{X_i}a_i, \pi^{Y_i}b_i)\frac{\tau_3(\alpha_3)\tau_3(\beta_3)}{\tau_3(\alpha_3\beta_3)}(-1)^{\frac{(\rho-1)}{m}(X_2Y_1+(Y_1+Y_2)X_3)} \\ &= (a_1^{Y_1}a_2^{Y_2}a_3^{Y_3}, \pi)_{\nu,m}\frac{\tau_3(\alpha_3)\tau_3(\beta_3)}{\tau_3(\alpha_3\beta_3)}(-1)^{\frac{(\rho-1)}{m}(X_2Y_1+(Y_1+Y_2)X_3)}, \end{aligned} \tag{2.4}$$

*where $\tau_3$ and $(-1)$ are both trivial whenever $m$ is odd.*

**PROOF OF THEOREM:** Once again, we begin by considering

$$\begin{aligned} dec_\nu(\alpha_3, \beta_3) &= \prod_{\xi\in\mu_m} \xi^{\int_{k_\nu{}^3\backslash 0} f(\alpha_3\mathcal{X})f(\xi\mathcal{X})\left(M(\mathcal{X})-M(\beta_3^{-1}\mathcal{X})\right)\mathrm{d}\mathcal{X}} \\ &=: \prod_{\xi\in\mu_m} \xi^{I(\xi)} =: \prod_{\xi\in\mu_m} \xi^{\sum_{i=1}^4 I_i(\xi)}. \end{aligned}$$

That is, recalling that $\hat{\mathcal{X}} := (x_1, x_2)^T$, we again decompose $k_\nu{}^3\backslash 0$ into four disjoint open sets,

$$\begin{aligned} A_1 &= \{(\hat{\mathcal{X}}, x_3)^T \in k_\nu{}^3\backslash 0 : \ |\hat{\mathcal{X}}|_\nu \geq |x_3|_\nu\ ,\ |\alpha_2\hat{\mathcal{X}}|_\nu \geq |\pi^{X_3}a_3x_3|_\nu\} \\ A_2 &= \{(\hat{\mathcal{X}}, x_3)^T \in k_\nu{}^3\backslash 0 : \ |\hat{\mathcal{X}}|_\nu \geq |x_3|_\nu\ ,\ |\alpha_2\hat{\mathcal{X}}|_\nu < |\pi^{X_3}a_3x_3|_\nu\} \\ A_3 &= \{(\hat{\mathcal{X}}, x_3)^T \in k_\nu{}^3\backslash 0 : \ |\hat{\mathcal{X}}|_\nu < |x_3|_\nu\ ,\ |\alpha_2\hat{\mathcal{X}}|_\nu \geq |\pi^{X_3}a_3x_3|_\nu\} \\ A_4 &= \{(\hat{\mathcal{X}}, x_3)^T \in k_\nu{}^3\backslash 0 : \ |\hat{\mathcal{X}}|_\nu < |x_3|_\nu\ ,\ |\alpha_2\hat{\mathcal{X}}|_\nu < |\pi^{X_3}a_3x_3|_\nu\}, \end{aligned}$$

and then calculate each of the integrals,

$$I_i(\xi) = \int_{A_i} f(\alpha_3\mathcal{X})f(\xi\mathcal{X})\big(M(\mathcal{X}) - M(\beta_3^{-1}\mathcal{X})\big)\mathrm{d}\mathcal{X}.$$

Considering the value of our function $f$ on each of our sets $A_i$ we find,

$$\begin{array}{lcll}
\mathcal{X} \in A_1 & \Rightarrow & f(\alpha_3\mathcal{X}) = f(\alpha_2\hat{\mathcal{X}}), & f(\xi\mathcal{X}) = f(\xi\hat{\mathcal{X}}) \\
\mathcal{X} \in A_2 & \Rightarrow & f(\alpha_3\mathcal{X}) = f(\pi^{X_3}a_3x_3), & f(\xi\mathcal{X}) = f(\xi\hat{\mathcal{X}}) \\
\mathcal{X} \in A_3 & \Rightarrow & f(\alpha_3\mathcal{X}) = f(\alpha_2\hat{\mathcal{X}}), & f(\xi\mathcal{X}) = f(\xi x_3) \\
\mathcal{X} \in A_4 & \Rightarrow & f(\alpha_3\mathcal{X}) = f(\pi^{X_3}a_3x_3), & f(\xi\mathcal{X}) = f(\xi x_3).
\end{array}$$

So, once again it shall be convenient to calculate these integrals in pairs.

**The integrals $I_1(\xi)$ and $I_4(\xi)$ :**

Considering the integrals $I_1(\xi)$ and $I_4(\xi)$ we find,

$$I_1(\xi) = \int_{A_1} f(\alpha_2\hat{\mathcal{X}})f(\xi\hat{\mathcal{X}})\big(M(\mathcal{X}) - M(\beta_3^{-1}\mathcal{X})\big)\mathrm{d}\hat{\mathcal{X}}\mathrm{d}x_3$$

$$I_4(\xi) = \int_{A_4} f(\pi^{X_3}a_3x_3)f(\xi x_3)\big(M(\mathcal{X}) - M(\beta_3^{-1}\mathcal{X})\big)\mathrm{d}\hat{\mathcal{X}}\mathrm{d}x_3.$$

Let us now define the sets,

$$A_1(\hat{\mathcal{X}}) = \{x_3 \in k_\nu : (\hat{\mathcal{X}}, x_3)^T \in A_1\}, \qquad A_4(x_3) = \{\hat{\mathcal{X}} \in {k_\nu}^2 : (\hat{\mathcal{X}}, x_3)^T \in A_4\},$$

then both $A_1(\hat{\mathcal{X}})$ and $A_4(x_3)$ are compact, open, $\mu_m$-invariant, neighbourhoods of zero in $k_\nu$ and ${k_\nu}^2$ respectively.

By restriction $M : A_1(\hat{\mathcal{X}}) \to \mathbb{Z}$ and $M : A_4(x_3) \to \mathbb{Z}$ are the characteristic functions of $A_1(\hat{\mathcal{X}}) \cap \mathfrak{O}_\nu$ and $A_4(x_3) \cap \mathfrak{O}_\nu^2$. Since both of these spaces are compact, open, $\mu_m$-invariant neighbourhoods of zero in $k_\nu$ and ${k_\nu}^2$ respectively, we have again satisfied the conditions of Lemma 1. Thus, for the integral $I_1(\xi)$ we are able to calculate,

$$\begin{aligned}
I_1(\xi) &= \int\limits_{{k_\nu}^2\backslash 0} f(\alpha_2\hat{\mathcal{X}})f(\xi\hat{\mathcal{X}}) \int\limits_{A_1(\hat{\mathcal{X}})} (M(\mathcal{X}) - M(\beta_3^{-1}\mathcal{X}))\mathrm{d}\hat{\mathcal{X}}\mathrm{d}x_3 \\
&= \int\limits_{{k_\nu}^2\backslash 0} f(\alpha_2\hat{\mathcal{X}})f(\xi\hat{\mathcal{X}}) \left\{\left(M(\hat{\mathcal{X}}) \int\limits_{A_1(\hat{\mathcal{X}})} M(x_3)\mathrm{d}x_3\right) - \left(M(\beta_2^{-1}\hat{\mathcal{X}}) \int\limits_{A_1(\hat{\mathcal{X}})} M(\pi^{-Y_3}b_3^{-1}x_3)\mathrm{d}x_3\right)\right\}\mathrm{d}\hat{\mathcal{X}} \\
&\equiv \int\limits_{{k_\nu}^2\backslash 0} f(\alpha_2\hat{\mathcal{X}})f(\xi\hat{\mathcal{X}})\,(M(\hat{\mathcal{X}}) - M(\beta_2^{-1}\hat{\mathcal{X}}))\mathrm{d}\hat{\mathcal{X}} \pmod{m}.
\end{aligned}$$

A similar calculation for $I_4(\xi)$ yields,

$$I_4(\xi) = \int_{k_\nu{}^2\backslash 0} f(\pi^{X_3}a_3x_3)f(\xi x_3)\left\{\left(M(x_3)\int_{A_4(x_3)} M(\hat{\mathcal{X}})\mathrm{d}\hat{\mathcal{X}}\right) - \left(M(\pi^{-Y_3}b_3^{-1}x_3)\int_{A_4(x_3)} M(\beta_2^{-1}\hat{\mathcal{X}})\mathrm{d}\hat{\mathcal{X}}\right)\right\}\mathrm{d}x_3$$

$$\equiv \int_{k_\nu\backslash 0} f(\pi^{X_3}a_3x_3)f(\xi x_3)\big(M(x_3) - M(\pi^{-Y_3}b_3^{-1}x_3)\big)\mathrm{d}x_3 \pmod{m}.$$

We conclude this section by noting that the integrals $I_1(\xi)$ and $I_4(\xi)$ have been found to satisfy,

$$\prod_{\xi\in\mu_m} \xi^{I_1(\xi)+I_4(\xi)} = dec_\nu(\alpha_2,\beta_2)\ dec_\nu(\pi^{X_3}a_3,\pi^{Y_3}b_3).$$

**The integrals $I_2(\xi)$ and $I_3(\xi)$:**

In order to prove our theorem it simply remains for us to calculate the integrals,

$$I_2(\xi) = \int_{A_2} f(\pi^{X_3}a_3x_3)f(\xi\hat{\mathcal{X}})\big(M(\mathcal{X}) - M(\beta_3^{-1}\mathcal{X})\big)\mathrm{d}\mathcal{X}$$

$$I_3(\xi) = \int_{A_3} f(\alpha_2\hat{\mathcal{X}})f(\xi x_3)\big(M(\mathcal{X}) - M(\beta_3^{-1}\mathcal{X})\big)\mathrm{d}\mathcal{X}.$$

To solve these integrals we note that there is an action of $\mu_m \oplus\ \mu_m$ on $k_\nu{}^2 \oplus k_\nu$ where

$$A_2,\quad A_3\quad\text{and}\quad \big(M(\mathcal{X}) - M(\beta_3^{-1}\mathcal{X})\big)\mathrm{d}\hat{\mathcal{X}}\mathrm{d}x_3$$

are again $\mu_m \oplus\ \mu_m$-invariant. Using this fact, as on page 33, we are able to calculate

$$m^2I_2(\xi) = \sum_{\xi_1,\xi_2\in\mu_m}\int_{A_2} f(\pi^{X_3}a_3\xi_1x_3)f(\xi\xi_2\hat{\mathcal{X}})\big(M(\mathcal{X}) - M(\beta_3^{-1}\mathcal{X})\big)\mathrm{d}\mathcal{X}$$

$$= \int_{A_2} \big(M(\mathcal{X}) - M(\beta_3^{-1}\mathcal{X})\big)\mathrm{d}\mathcal{X}$$

$$m^2I_3(\xi) = \sum_{\xi_1,\xi_2\in\mu_m}\int_{A_3} f(\alpha_2\xi_1\hat{\mathcal{X}})f(\xi\xi_2x_3)\big(M(\mathcal{X}) - M(\beta_3^{-1}\mathcal{X})\big)\mathrm{d}\mathcal{X}$$

$$= \int_{A_3} \big(M(\mathcal{X}) - M(\beta_3^{-1}\mathcal{X})\big)\mathrm{d}\mathcal{X}.$$

Once again we find that the integrals $I_2(\xi) := I_2$ and $I_3(\xi) := I_3$ are independent of $\xi$. This allows us to concentrate on calculating,

$$\prod_{\xi\in\mu_m} \xi^{I_2(\xi)+I_3(\xi)} = (-1)^{I_2+I_3} = (-1)^{I_2-I_3},$$

where, if $m$ is odd, this expression is known to be trivial.

Considering the proof of the previous theorem we define the set,

$$S_3 = \{\mathcal{X} \in {k_\nu}^3 \backslash 0 : |\hat{\mathcal{X}}|_\nu \geq |x_3|_\nu\},$$

which allows us to write

$$A_2 - A_3 = (A_2 + A_1) - (A_3 + A_1) = S_3 - \alpha_3^{-1} S_3.$$

Therefore the difference of the integrals may now be written as,

$$\begin{aligned} I_2 - I_3 &= \frac{1}{m^2}\left\{\int_{S_3} - \int_{\alpha_3^{-1}S_3}\right\}\Big(M(\mathcal{X}) - M(\beta_3^{-1}\mathcal{X})\Big)\mathrm{d}\mathcal{X} \qquad (2.5) \\ &= \frac{1}{m^2}\int_{S_3}\Big(M(\mathcal{X}) - M(\beta_3^{-1}\mathcal{X})\Big)\mathrm{d}\mathcal{X} - |\det(\alpha_3)|_\nu^{-1}\int_{S_3}\Big(M(\alpha_3^{-1}\mathcal{X}) - M((\alpha_3\beta_3)^{-1}\mathcal{X})\Big)\mathrm{d}\mathcal{X}. \end{aligned}$$

We now define the function $J_3$ on the torus in $\mathrm{GL}_3(k_\nu)$ by,

$$J_3(\alpha_3) = \int_{S_3}(M(\mathcal{X}) - M(\alpha_3^{-1}\mathcal{X}))\mathrm{d}\mathcal{X}.$$

Then equation (2.5) may be expressed as,

$$I_2 - I_3 = \frac{1}{m^2}\Big(J_3(\beta_3) + |\det(\alpha_3)|_\nu^{-1} J_3(\alpha_3) - |\det(\alpha_3)|_\nu^{-1} J_3(\alpha_3\beta_3)\Big).$$

Since we know (Page 761, [3]) that the function $J_3$ satisfies,

$$J_3(\alpha_3) \equiv 1 - |\pi^{X_1+X_2}a_1a_2|_\nu \pmod{m^2},$$

we are free to define the new function $\jmath_3$ by,

$$\jmath_3(\alpha_3) := (-1)^{\frac{1}{m^2}\left(J_3(\alpha_3) - (1-|\pi^{X_1+X_2}a_1a_2|_\nu)\right)},$$

where the right hand side of this equation is a well defined element of $\mu_m$.

Substituting for this function $\jmath_3$ and performing a similar calculation to that on page 34 we eventually find that,

$$(-1)^{I_2+I_3} = \frac{\jmath_3(\alpha_3)\jmath_3(\beta_3)}{\jmath_3(\alpha_3\beta_3)}(-1)^{\frac{(\rho-1)}{m}(Y_1+Y_2)X_3}.$$

Using all that we have discovered in this chapter, on the torus $T$ in $\mathrm{GL}_3(k_\nu)$, we may conclude that the cocycle $dec_\nu$ satisfies,

$$
\begin{aligned}
dec_\nu(\alpha_3,\beta_3) &= dec_\nu(\alpha_2,\beta_2)\, dec_\nu(\pi^{X_3}a_3,\pi^{Y_3}b_3).\frac{\jmath_3(\alpha_3)\jmath_3(\beta_3)}{\jmath_3(\alpha_3\beta_3)}(-1)^{\frac{(p-1)}{m}(Y_1+Y_2)X_3} \\
&= dec_\nu(\pi^{X_1}a_1,\pi^{Y_1}b_1)\, dec_\nu(\pi^{X_2}a_2,\pi^{Y_2}b_2)\frac{\jmath_2(\alpha_2)\jmath_2(\beta_2)}{\jmath_2(\alpha_2\beta_2)}(-1)^{\frac{(p-1)}{m}Y_1X_2} \\
&\quad\times\ dec_\nu(\pi^{X_3}a_3,\pi^{Y_3}b_3)\frac{\jmath_3(\alpha_3)\jmath_3(\beta_3)}{\jmath_3(\alpha_3\beta_3)}(-1)^{\frac{(p-1)}{m}(Y_1+Y_2)X_3} \\
&= (a_1^{Y_1}a_2^{Y_2}a_3^{Y_3},\pi)_{\nu,m}\frac{\tau_3(\alpha_3)\tau_3(\beta_3)}{\tau_3(\alpha_3\beta_3)}(-1)^{\frac{(p-1)}{m}(Y_1X_2+(Y_1+Y_2)X_3)},
\end{aligned}
$$

where $\tau_3(\alpha_3)=\jmath_2(\alpha_2)\jmath_3(\alpha_3)$ and the functions $\jmath_i$ are as previously described.

□

## 2.4 The cocycle $dec_\nu$ on $T \subset \mathrm{GL}_n(k_\nu)$

**Theorem 2.4.1** *Let $k_\nu$ be a local field with valuation $\nu$. Then there exists an explicit function,*

$$\tau_n : T \longrightarrow \{1,-1\},$$

*such that, for each $\alpha_n$, $\beta_n \in T \subset GL_n(k_\nu)$ the cocycle $dec_\nu$ satisfies,*

$$
\begin{aligned}
dec_\nu(\alpha_n,\beta_n) &= \prod_{i=1}^{n} dec_\nu(\pi^{X_i}a_i,\pi^{Y_i}b_i)\prod_{i<j}(-1)^{\frac{(p-1)}{m}Y_iX_j}\frac{\tau_n(\alpha_n)\tau_n(\beta_n)}{\tau_n(\alpha_n\beta_n)} \\
&= \prod_{i=1}^{n}(a_i^{Y_i},\pi)_{\nu,m}\prod_{i<j}(-1)^{\frac{(p-1)}{m}Y_iX_j}\frac{\tau_n(\alpha_n)\tau_n(\beta_n)}{\tau_n(\alpha_n\beta_n)} \qquad (2.6)
\end{aligned}
$$

*where $\tau_n$ and $(-1)$ are both trivial whenever $m$ is odd.*

PROOF OF THEOREM:

Considering the previous two theorems it should be clear that we intend to prove this result by induction. Therefore, let us assume that our result is indeed true in $n-1$ dimensions.

Let us now consider the situation in $n$ dimensions. By definition the cocycle satisfies,

$$dec_\nu(\alpha_n,\beta_n) = \prod_{\xi\in\mu_m}\xi^{\int_{k_\nu{}^n\setminus 0} f(\alpha_n\mathcal{X})f(\xi\mathcal{X})\left(M(\mathcal{X})-M(\beta_n^{-1}\mathcal{X})\right)\mathrm{d}\mathcal{X}} =: \prod_{\xi\in\mu_m}\xi^{I(\xi)},$$

where the integral $I(\xi)$ is defined to be,

$$I(\xi) = \int_{k_\nu{}^n\backslash 0} f(\alpha_n \mathcal{X}) f(\xi\mathcal{X})(M(\mathcal{X}) - M(\beta_n^{-1}\mathcal{X}))\mathrm{d}\mathcal{X}.$$

Let us now decompose $k_\nu{}^n\backslash 0$ into the four disjoint open sets,

$$\begin{aligned}
A_1 &= \{(\hat{\mathcal{X}}, x_n)^T \in k_\nu{}^n\backslash 0 : \ |\hat{\mathcal{X}}|_\nu \geq |x_n|_\nu \ , \ |\alpha_{n-1}\hat{\mathcal{X}}|_\nu \geq |\pi^{X_n} a_n x_n|_\nu\} \\
A_2 &= \{(\hat{\mathcal{X}}, x_n)^T \in k_\nu{}^n\backslash 0 : \ |\hat{\mathcal{X}}|_\nu \geq |x_n|_\nu \ , \ |\alpha_{n-1}\hat{\mathcal{X}}|_\nu < |\pi^{X_n} a_n x_n|_\nu\} \\
A_3 &= \{(\hat{\mathcal{X}}, x_n)^T \in k_\nu{}^n\backslash 0 : \ |\hat{\mathcal{X}}|_\nu < |x_n|_\nu \ , \ |\alpha_{n-1}\hat{\mathcal{X}}|_\nu \geq |\pi^{X_n} a_n x_n|_\nu\} \\
A_4 &= \{(\hat{\mathcal{X}}, x_n)^T \in k_\nu{}^n\backslash 0 : \ |\hat{\mathcal{X}}|_\nu < |x_n|_\nu \ , \ |\alpha_{n-1}\hat{\mathcal{X}}|_\nu < |\pi^{X_n} a_n x_n|_\nu\},
\end{aligned}$$

where, for $1 \leq i \leq 4$, the integrals $I_i(\xi)$ are defined by,

$$I_i(\xi) = \int_{A_i} f(\alpha_n \mathcal{X}) f(\xi\mathcal{X})(M(\mathcal{X}) - M(\beta_n^{-1}\mathcal{X}))\mathrm{d}\mathcal{X}.$$

Considering the value of the function $f$ on each of the sets $A_i$ we find that,

$$\begin{array}{llll}
\mathcal{X} \in A_1 & \Rightarrow & f(\alpha_n\mathcal{X}) = f(\alpha_{n-1}\hat{\mathcal{X}}), & f(\xi\mathcal{X}) = f(\xi\hat{\mathcal{X}}) \\
\mathcal{X} \in A_2 & \Rightarrow & f(\alpha_n\mathcal{X}) = f(\pi^{X_n} a_n x_n), & f(\xi\mathcal{X}) = f(\xi\hat{\mathcal{X}}) \\
\mathcal{X} \in A_3 & \Rightarrow & f(\alpha_n\mathcal{X}) = f(\alpha_{n-1}\hat{\mathcal{X}}), & f(\xi\mathcal{X}) = f(\xi x_n) \\
\mathcal{X} \in A_4 & \Rightarrow & f(\alpha_n\mathcal{X}) = f(\pi^{X_n} a_n x_n), & f(\xi\mathcal{X}) = f(\xi x_n).
\end{array}$$

As with the previous sections we shall proceed by calculating the integrals $I_i(\xi)$ in pairs.

**The integrals $I_1(\xi)$ and $I_4(\xi)$:**

Using the results on the previous page, for the integrals $I_1(\xi)$ and $I_4(\xi)$, we find that we have

$$\begin{aligned}
I_1(\xi) &= \int_{A_1} f(\alpha_{n-1}\hat{\mathcal{X}}) f(\xi\hat{\mathcal{X}})(M(\mathcal{X}) - M(\beta_n^{-1}\mathcal{X}))\mathrm{d}\hat{\mathcal{X}}\mathrm{d}x_n \\
I_4(\xi) &= \int_{A_4} f(\pi^{X_n} a_n x_n) f(\xi x_n)(M(\mathcal{X}) - M(\beta_n^{-1}\mathcal{X}))\mathrm{d}\hat{\mathcal{X}}\mathrm{d}x_n.
\end{aligned}$$

Let us now define the sets $A_1(\hat{\mathcal{X}})$ and $A_4(x_n)$ by,

$$A_1(\hat{\mathcal{X}}) = \{x_n \in k_\nu : (\hat{\mathcal{X}}, x_n)^T \in A_1\}, \qquad A_4(x_n) = \{\hat{\mathcal{X}} \in k_\nu{}^{n-1} : (\hat{\mathcal{X}}, x_n)^T \in A_4\}.$$

Then by restriction $M : A_1(\hat{\mathcal{X}}) \to \mathbb{Z}$ and $M : A_4(x_n) \to \mathbb{Z}$ are the characteristic functions of $A_1(\hat{\mathcal{X}}) \cap \mathfrak{O}_\nu$ and $A_4(x_n) \cap \mathfrak{O}_\nu^{n-1}$.

Therefore we are once again able to use Lemma 1 to calculate,

$$
\begin{aligned}
I_1(\xi) &= \int\limits_{{k_\nu}^{n-1}\backslash 0} f(\alpha_{n-1}\hat{\mathcal{X}})f(\xi\hat{\mathcal{X}}) \int\limits_{A_1(\hat{\mathcal{X}})} \Big(M(\mathcal{X}) - M(\beta_n^{-1}\mathcal{X})\Big)\, \mathrm{d}\hat{\mathcal{X}}\mathrm{d}x_n \\
&= \int\limits_{{k_\nu}^{n-1}\backslash 0} f(\alpha_{n-1}\hat{\mathcal{X}})f(\xi\hat{\mathcal{X}}) \left\{ \Big(M(\hat{\mathcal{X}}) \int\limits_{A_1(\hat{\mathcal{X}})} M(x_n)\mathrm{d}x_n\Big) - \Big(M(\beta_{n-1}^{-1}\hat{\mathcal{X}}) \int\limits_{A_1(\hat{\mathcal{X}})} M(\pi^{-Y_n}b_n^{-1}x_n)\mathrm{d}x_n\Big)\right\}\mathrm{d}\hat{\mathcal{X}} \\
&\equiv \int\limits_{{k_\nu}^{n-1}\backslash 0} f(\alpha_{n-1}\hat{\mathcal{X}})f(\xi\hat{\mathcal{X}})\, (M(\hat{\mathcal{X}}) - M(\beta_{n-1}^{-1}\hat{\mathcal{X}}))\mathrm{d}\hat{\mathcal{X}} \qquad (\mathrm{mod}\ m).
\end{aligned}
$$

and similarly for $I_4(\xi)$ we find,

$$
\begin{aligned}
I_4(\xi) &= \int\limits_{k_\nu\backslash 0} f(\pi^{X_n}a_nx_n)f(\xi x_n) \left\{ \Big(M(x_n) \int\limits_{A_4(x_n)} M(\hat{\mathcal{X}})\mathrm{d}\hat{\mathcal{X}}\Big) - \Big(M(\pi^{-Y_n}b_n^{-1}x_n) \int\limits_{A_4(x_n)} M(\beta_{n-1}^{-1}\hat{\mathcal{X}})\mathrm{d}\hat{\mathcal{X}}\Big)\right\}\mathrm{d}x_n \\
&\equiv \int\limits_{k_\nu\backslash 0} f(\pi^{X_n}a_nx_n)f(\xi x_n)\big(M(x_n) - M(\pi^{-Y_n}b_n^{-1}x_n)\big)\, \mathrm{d}x_n \qquad (\mathrm{mod}\ m).
\end{aligned}
$$

Finally we are able to conclude that,

$$\prod_{\xi\in\mu_m} \xi^{I_1(\xi)+I_4(\xi)} = dec_\nu(\alpha_{n-1},\beta_{n-1})\, dec_\nu(\pi^{X_n}a_n, \pi^{Y_n}b_n).$$

**The integrals $I_2(\xi)$ and $I_3(\xi)$:**

For the integrals $I_2(\xi)$ and $I_3(\xi)$ we find that we have,

$$
\begin{aligned}
I_2(\xi) &= \int_{A_2} f(\pi^{X_n}a_nx_n)f(\xi\hat{\mathcal{X}})\big(M(\mathcal{X}) - M(\beta_n^{-1}\mathcal{X})\big)\mathrm{d}\mathcal{X} \\
I_3(\xi) &= \int_{A_3} f(\alpha_{n-1}\hat{\mathcal{X}})f(\xi x_n)\big(M(\mathcal{X}) - M(\beta_n^{-1}\mathcal{X})\big)\mathrm{d}\mathcal{X}.
\end{aligned}
$$

In this case we must use the action of $\mu_m \oplus \mu_m$ on ${k_\nu}^{n-1} \oplus k_\nu$. Once again we find that the sets,

$$A_2, \quad A_3 \quad \text{and} \quad \big(M(\mathcal{X}) - M(\beta_n^{-1}\mathcal{X})\big)\, \mathrm{d}\hat{\mathcal{X}}\mathrm{d}x_n,$$

are all $\mu_m \oplus \mu_m$-invariant.

Therefore, since $f$ is fundamental, we are able to deduce that these integrals satisfy,

$$\begin{aligned} m^2 I_2(\xi) &= \sum_{\xi_1,\xi_2\in\mu_m} \int_{A_2} f(\pi^{X_n} a_n \xi_1 x_n) f(\xi\xi_2 \hat{\mathcal{X}})\big(M(\mathcal{X}) - M(\beta_n^{-1}\mathcal{X})\big)\, \mathrm{d}\mathcal{X} \\ &= \int_{A_2} (M(\mathcal{X}) - M(\beta_n^{-1}\mathcal{X}))\, \mathrm{d}\mathcal{X} \\ m^2 I_3(\xi) &= \sum_{\xi_1,\xi_2\in\mu_m} \int_{A_3} f(\alpha_{n-1}\xi_1\hat{\mathcal{X}}) f(\xi\xi_2 x_n)\big(M(\mathcal{X}) - M(\beta_n^{-1}\mathcal{X})\big)\, \mathrm{d}\mathcal{X} \\ &= \int_{A_3} (M(\mathcal{X}) - M(\beta_n^{-1}\mathcal{X}))\, \mathrm{d}\mathcal{X}. \end{aligned}$$

Both integrals $I_2(\xi) := I_2$ and $I_3(\xi) := I_3$ are again independent of $\xi$. So again we need only concern ourselves with finding the value of the expression,

$$\prod_{\xi\in\mu_m} \xi^{I_2(\xi)+I_3(\xi)} = (-1)^{I_2+I_3} = (-1)^{I_2-I_3},$$

where $(-1)$ is trivial whenever $m$ is odd.

By defining the set $S_n = \{\mathcal{X} \in k_\nu{}^n\backslash 0 : |\hat{\mathcal{X}}|_\nu \geq |x_n|_\nu\}$ and then noting that,

$$A_2 - A_3 = (A_2 + A_1) - (A_3 + A_1) = S_n - \alpha_n^{-1} S_n,$$

we are able to express the difference of the two integrals as,

$$\begin{aligned} I_2 - I_3 &= \frac{1}{m^2}\left\{\int_{S_n} - \int_{\alpha_n^{-1}S_n}\right\} (M(\mathcal{X}) - M(\beta_n^{-1}\mathcal{X}))\, \mathrm{d}\mathcal{X} \\ &= \frac{1}{m^2}\int_{S_n}\Big(M(\mathcal{X}) - M(\beta_n^{-1}\mathcal{X})\Big)\mathrm{d}\mathcal{X} - |\det(\alpha_n)|_\nu^{-1}\int_{S_n}\Big(M(\alpha_n^{-1}\mathcal{X}) - M((\alpha_n\beta_n)^{-1}\mathcal{X})\Big)\mathrm{d}\mathcal{X} \\ &= \frac{1}{m^2}\Big(J_n(\beta_n) + |\det(\alpha_n)|_\nu^{-1} J_n(\alpha_n) - |\det(\alpha_n)|_\nu^{-1} J_n(\alpha_n\beta_n)\Big), \end{aligned}$$

where the function $J_n$ on the torus in $\mathrm{GL}_n(k_\nu)$ is defined by,

$$J_n(\alpha_n) = \int_{S_n} (M(\mathcal{X}) - M(\alpha_n^{-1}\mathcal{X}))\, \mathrm{d}\mathcal{X}$$

and satisfies,

$$\equiv 1 - |\pi^{X_1+X_2+\ldots+X_{n-1}} a_1 a_2 \ldots a_{n-1}|_\nu \pmod{m^2}.$$

Having defined the related function,

$$\jmath_n(\alpha_n) := (-1)^{\frac{1}{m^2}\left(J_n(\alpha_n) - (1 - |\pi^{X_1+X_2+\ldots+X_{n-1}} a_1 a_2 \ldots a_{n-1}|_\nu)\right)},$$

it swiftly follows that,

$$(-1)^{I_2+I_3} = \frac{\jmath_n(\alpha_n)\jmath_n(\beta_n)}{\jmath_n(\alpha_n\beta_n)} (-1)^{\frac{(\rho-1)}{m}(Y_1+Y_2+\ldots+Y_{n-1})X_n}.$$

Finally, using our inductive hypothesis we have proven that on the torus $T$ in $\mathrm{GL}_n(k_\nu)$ the cocycle $dec_\nu$ satisfies,

$$\begin{aligned} dec_\nu(\alpha_n, \beta_n) &= dec_\nu(\alpha_{n-1}, \beta_{n-1})\, dec_\nu(\pi^{X_n} a_n, \pi^{Y_n} b_n) \\ &\quad \times \frac{\jmath_n(\alpha_n)\jmath_n(\beta_n)}{\jmath_n(\alpha_n\beta_n)}(-1)^{\frac{(p-1)}{m}(Y_1+Y_2+\ldots+Y_{n-1})X_n} \\ &= \prod_{i=1}^{n-1} dec_\nu(\pi^{X_i} a_i, \pi^{Y_i} b_i) \frac{\tau_{n-1}(\alpha_{n-1})\tau_{n-1}(\beta_{n-1})}{\tau_{n-1}(\alpha_{n-1}\beta_{n-1})} \prod_{i<j}(-1)^{\frac{(p-1)}{m}Y_iX_j} \\ &\quad \times\, dec_\nu(\pi^{X_n} a_n, \pi^{Y_n} b_n) \frac{\jmath_n(\alpha_n)\jmath_n(\beta_n)}{\jmath_n(\alpha_n\beta_n)}(-1)^{\frac{(p-1)}{m}(Y_1+Y_2+\ldots+Y_{n-1})X_n}. \end{aligned}$$

Finally, since we have

$$\tau_{n-1}(\alpha_{n-1}).\jmath_n(\alpha_n) = \jmath_2(\alpha_2)\jmath_3(\alpha_3)\ldots\jmath_{n-1}(\alpha_{n-1}).\jmath_n(\alpha_n) = \tau_n(\alpha_n),$$

we do indeed find,

$$\begin{aligned} dec_\nu(\alpha_n, \beta_n) &= \prod_{i=1}^{n} dec_\nu(\pi^{X_i} a_i, \pi^{Y_i} b_i) \prod_{i<j}(-1)^{\frac{(p-1)}{m}Y_iX_j} \frac{\tau_n(\alpha_n)\tau_n(\beta_n)}{\tau_n(\alpha_n\beta_n)} \\ &= \prod_{i=1}^{n}(a_i^{Y_i}, \pi)_{\nu,m} \prod_{i<j}(-1)^{\frac{(p-1)}{m}Y_iX_j} \frac{\tau_n(\alpha_n)\tau_n(\beta_n)}{\tau_n(\alpha_n\beta_n)}, \end{aligned} \tag{2.7}$$

and hence our result is true for any dimension $n$.

□

# Chapter 3

# The Coboundaries $\partial\tau_n$

In this chapter we shall investigate the coboundaries $\partial\tau_n$, associated with the cocycle $dec_\nu \in Z^2(\mathrm{GL}_n(k_\nu), \mu_m)$, when the number of roots of unity in $\mu_m$ is even. In the previous chapter we had found that on the torus $T \subset \mathrm{GL}_n(k_\nu)$, the cocycle satisfies

$$dec_\nu(\alpha_n, \beta_n) = \left(\prod_{i=1}^{n} dec_\nu(\pi^{X_i}a_i, \pi^{Y_i}b_i)\right)\left(\prod_{i<j}(-1)^{\frac{(\rho-1)}{m}Y_iX_j}\right)\frac{\tau_n(\alpha_n)\tau_n(\beta_n)}{\tau_n(\alpha_n\beta_n)}, \tag{3.1}$$

where,

$$\tau_n(\alpha_n) = \jmath_2(\alpha_2)\jmath_3(\alpha_3)\dots\jmath_n(\alpha_n),$$

is trivial when $m$ is odd and where the function $\jmath_n$ is given by,

$$\jmath_n(\alpha_n) = (-1)^{\frac{1}{m^2}\left(J_n(\alpha_n) - (1 - |\pi^{X_1+\dots+X_{n-1}}a_1\dots a_{n-1}|_\nu)\right)}.$$

## 3.1 The function $\kappa_n$

**Definition:** We define the function $\kappa_n$ on $\mathrm{GL}_n(k_\nu)$ by,

$$\kappa_n(\alpha_n) = \int_{S_n} M(\alpha_n^{-1}\mathcal{X})\mathrm{d}\mathcal{X}$$

where, $$S_n = \{(\hat{\mathcal{X}}, x_n)^T \in {k_\nu}^n\backslash 0 : \max\{|x_1|_\nu, \dots, |x_{n-1}|_\nu\} \geq |x_n|_\nu\}.$$

Having made this definition we see that we are able to write,

$$J_n(\alpha_n) = \int_{S_n} \left(M(\mathcal{X}) - M(\alpha_n^{-1}\mathcal{X})\right)\mathrm{d}\mathcal{X} = \kappa_n(I_n) - \kappa_n(\alpha_n).$$

Therefore, throughout the next few sections we shall concentrate on finding the value of $\kappa_n(\alpha_n)$ for any $\alpha_n \in T$.

### 3.1.1 Values of the function $\kappa_n(I_n)$

In order to see how to proceed to the rest of the torus in $\mathrm{GL}_n(k_\nu)$ we shall first consider $\kappa_n$ on the identity in each dimension $n$. So we shall begin by calculating,

$$\kappa_n(I_n) = \int_{S_n} M(\mathcal{X})\mathrm{d}\mathcal{X}.$$

Since the function $M : k_\nu \mapsto \mathbb{Z}$ is the characteristic function of $\mathfrak{O}_\nu$ we find that $\kappa_n(I_n)$ is simply the measure of the set $(S_n \cap \mathfrak{O}_\nu^n)$.

We start by considering the cases of $n = 2$ and $n = 3$ as these we are able to illustrate in diagrams and calculate explicitly. We shall then move onto the general case, extending our work to any dimension $n$.

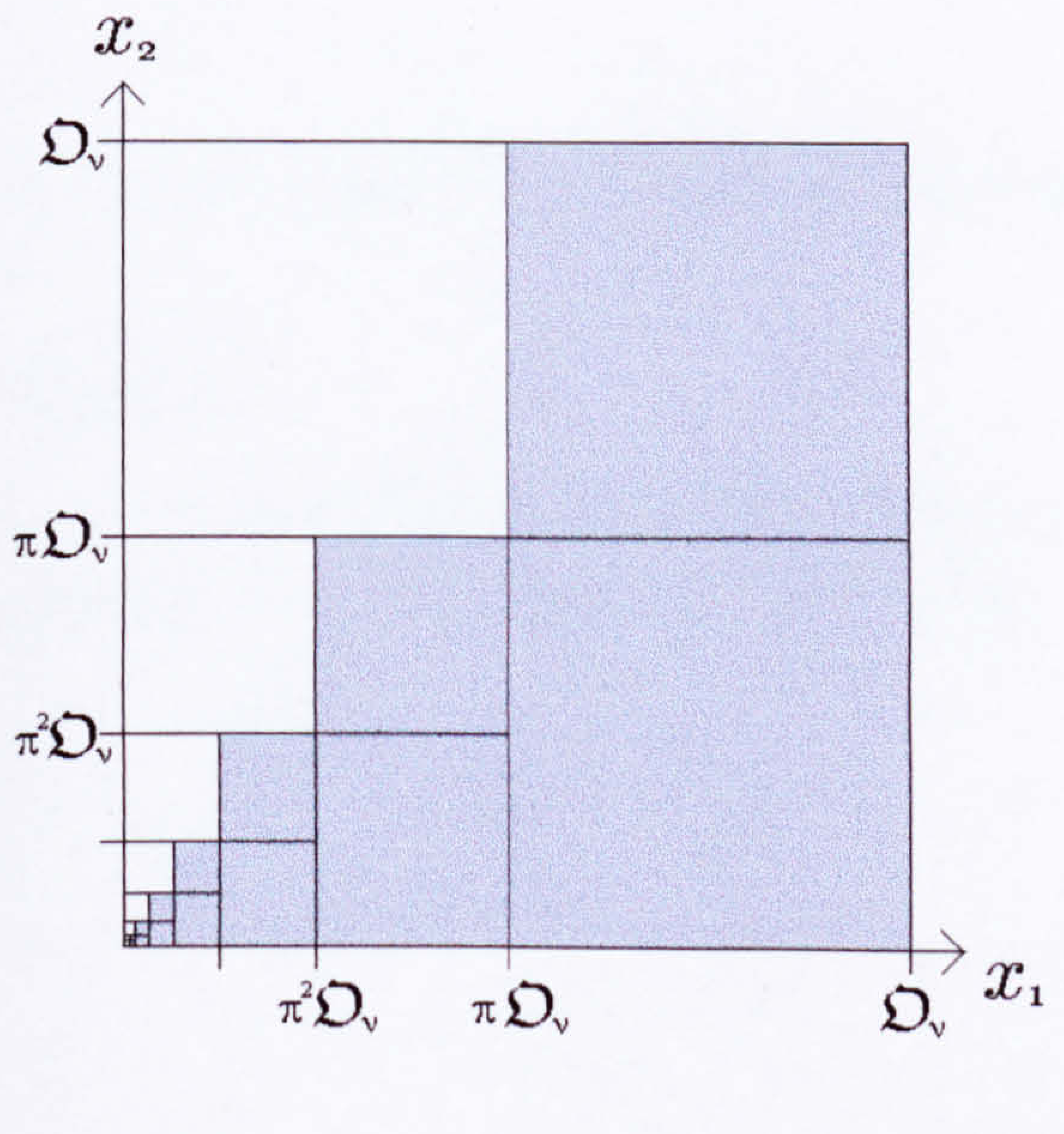
**Theorem 3.1.1** *As in the previous chapters let us define $\rho = p^f = |\pi|_\nu^{-1}$. Then on $GL_2(k_\nu)$ the function $\kappa_2$ satisfies,*

$$\kappa_2(I_2) = 1 - \frac{(\rho - 1)}{(\rho^2 - 1)}.$$

**PROOF OF THEOREM:**

As we said, we may graphically represent the measure of the set $(S_2 \cap \mathfrak{O}_\nu^2)$ and therefore the value of the function $\kappa_2(I_2)$ in a diagram.

We may consider the space $\mathfrak{O}_\nu^2$ as a square of area 1 with $(S_2 \cap \mathfrak{O}_\nu^2)$ being the area represented by the shaded region below. Considering the work to follow, we calculate this shaded area by taking the area of the whole square and subtracting off the area of the unshaded region.




This area is calculated to be,

$$\begin{aligned}\kappa_2(I_2) &= 1 - \sum_{k=0}^{\infty} \frac{1}{\rho^{(k+1)}}\left(\frac{1}{\rho^k} - \frac{1}{\rho^{k+1}}\right) \\ &= 1 - \frac{(\rho - 1)}{(\rho^2 - 1)}.\end{aligned}$$

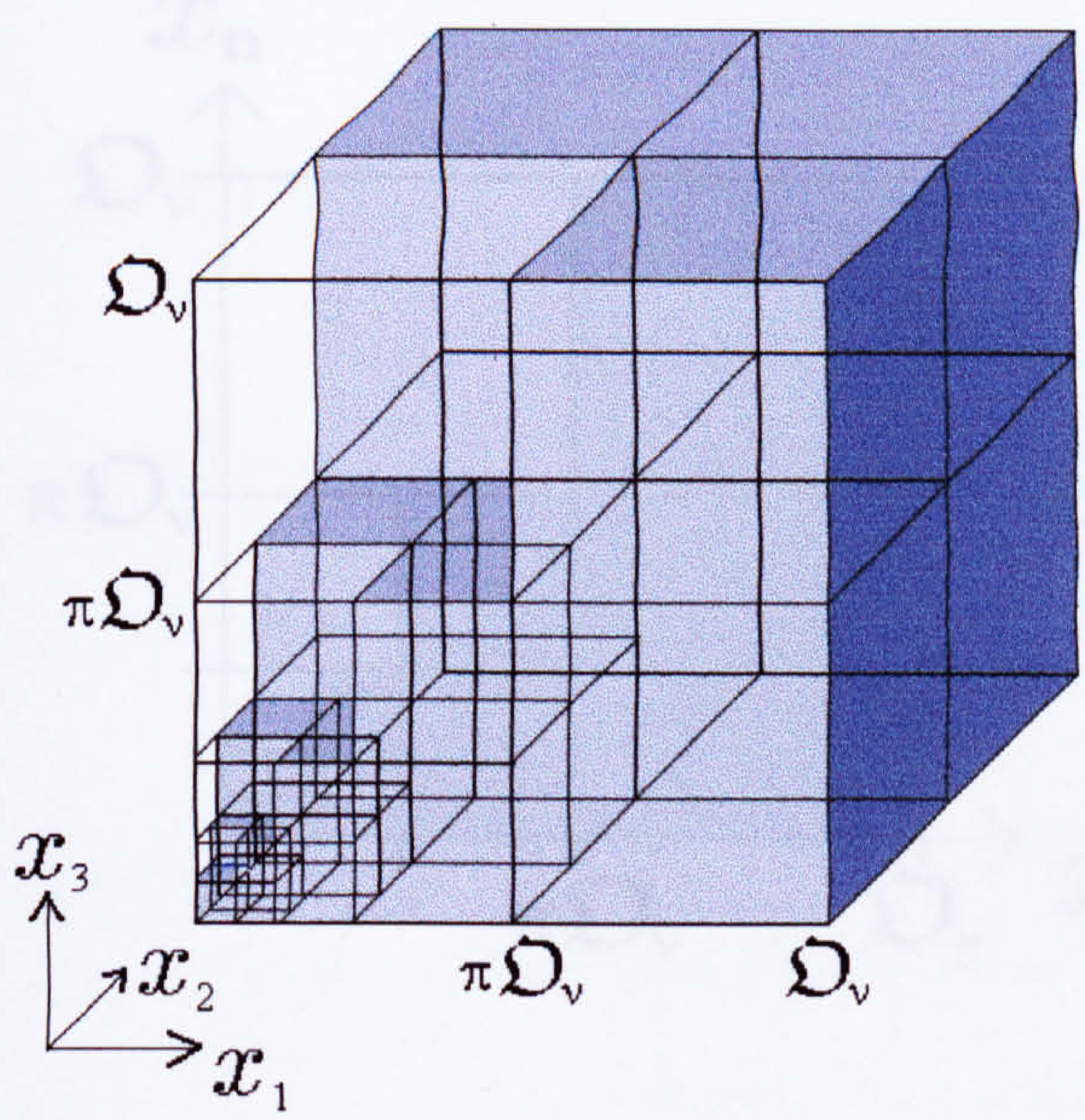
□

**Theorem 3.1.2** *On $GL_3(k_\nu)$ the function $\kappa_3$ satisfies,*

$$\kappa_3(I_3) = 1 - \frac{(\rho - 1)}{(\rho^3 - 1)}.$$

**PROOF OF THEOREM:**

Once again we may graphically represent the measure of the set $(S_3 \cap \mathfrak{O}_\nu^3)$. In three dimensions we consider the space $\mathfrak{O}_\nu^3$ as a cube of volume 1 where $(S_3 \cap \mathfrak{O}_\nu^3)$ is again represented by the shaded region. We calculate the volume of this region by considering the volume of the whole cube minus that of the unshaded region.




This volume is then given by,

$$\begin{aligned}\kappa_3(I_3) &= 1 - \sum_{k=0}^{\infty} \frac{1}{\rho^{2(k+1)}} \left( \frac{1}{\rho^k} - \frac{1}{\rho^{k+1}} \right) \\ &= 1 - \frac{(\rho - 1)}{(\rho^3 - 1)}.\end{aligned}$$

□

We now begin to have some intuitive idea of how to extend to any dimension $n$.

**Remark:**

It is worth pointing out that, since $\rho = |\pi|_\nu^{-1}$, the formulae for the function $\kappa$ are reminiscent of local factors of L-functions.

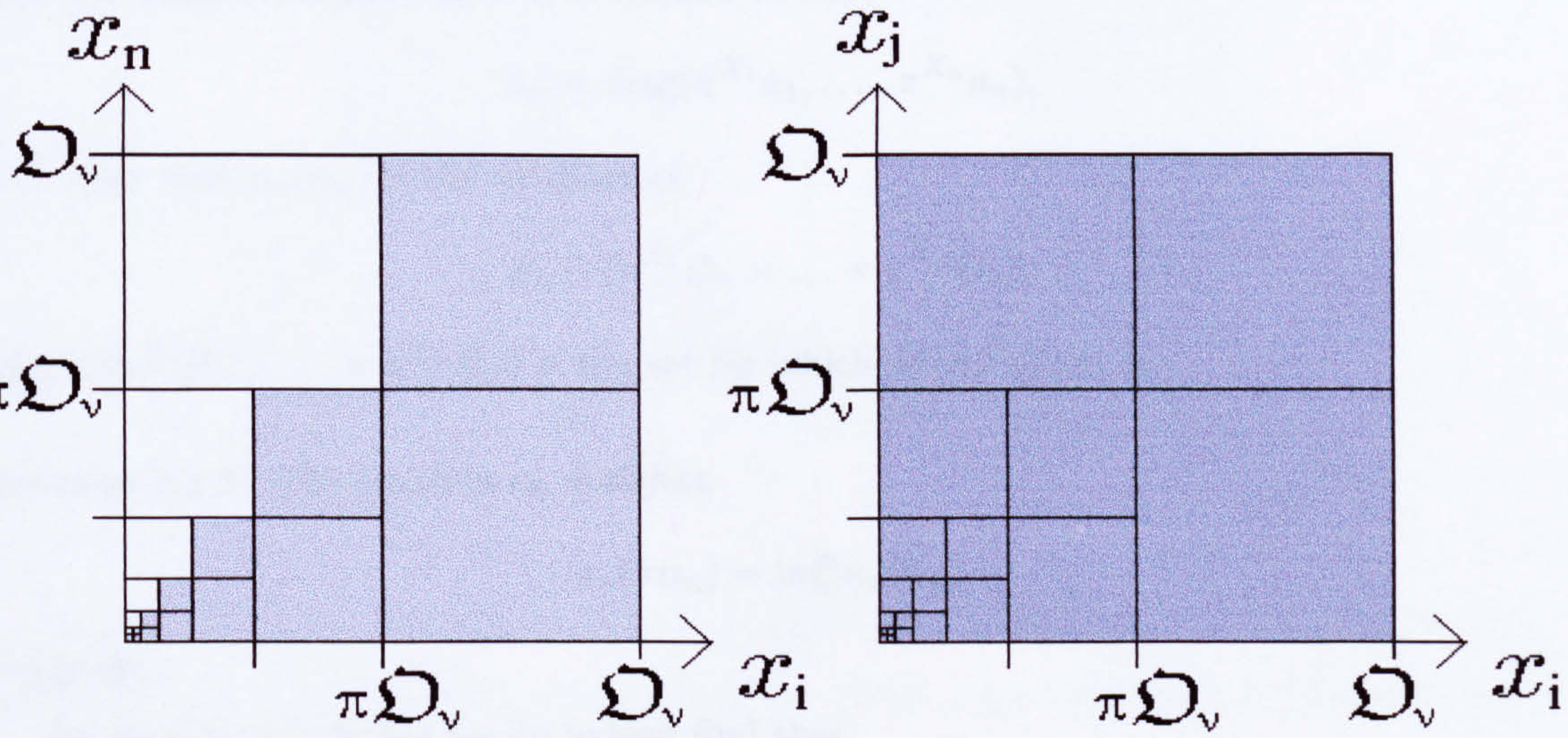
**Theorem 3.1.3** *On $GL_n(k_\nu)$ the function $\kappa_n$ satisfies,*

$$\kappa_n(I_n) = 1 - \frac{(\rho - 1)}{(\rho^n - 1)}.$$

**PROOF OF THEOREM:**

For the general case of any $n$ we are looking for the volume of $(S_n \cap \mathfrak{O}_\nu^n)$. We calculate this as the volume of an $n$-dimensional cube minus some unshaded region. Although we cannot draw this we may project the volume onto each pair of dimensions.

Having done so we find,




Considering these two diagrams we may deduce that as we move from $S_{n-1}$ to $S_n$ we change the volume of the unshaded region, the volume to be subtracted, by a multiple of $1/\rho^{(k+1)}$.

Therefore we may conclude that,

$$\begin{aligned}\kappa_n(I_n) &= 1 - \sum_{k=0}^{\infty} \frac{1}{\rho^{(n-1)(k+1)}} \left( \frac{1}{\rho^k} - \frac{1}{\rho^{k+1}} \right) \\ &= 1 - \frac{(\rho - 1)}{(\rho^n - 1)}.\end{aligned}$$

□

### 3.1.2 Values of the function $\kappa_n(\alpha_n)$

In this section we shall extend our knowledge of $\kappa_n$ to the whole of the torus in $\mathrm{GL}_n(k_\nu)$. We know that the function $\kappa_n$ satisfies,

$$\kappa_n(\alpha_n) = \int_{S_n} M(\alpha_n^{-1}\mathcal{X})\mathrm{d}\mathcal{X},$$

and again we may consider this to be the measure of some set. Having reminded ourselves that the diagonal matrix $\alpha_n \in T$ is defined to be,

$$\alpha_n = \mathrm{diag}(\pi^{X_1}a_1, \dots, \pi^{X_n}a_n),$$

it is clear that $\kappa_n(\alpha_n)$ is the measure of

$$S_n \cap (\pi^{X_1}\mathfrak{O}_\nu \times \dots \times \pi^{X_n}\mathfrak{O}_\nu),$$

where $(\pi^{X_1}\mathfrak{O}_\nu \times \dots \times \pi^{X_n}\mathfrak{O}_\nu)$ is the set for which $M(\alpha_n^{-1}\mathcal{X}) = 1$.

**Lemma 3.1.1** *The function $\kappa_n$ satisfies,*

$$\kappa_n(\pi\alpha_n) = |\pi|_\nu^n \kappa_n(\alpha_n).$$

**PROOF:**

By simply calculating we do indeed find that,

$$\begin{aligned}\kappa_n(\pi\alpha_n) &= \int_{S_n} M(\pi^{-1}\alpha_n^{-1}\mathcal{X})\mathrm{d}\mathcal{X}\\ &= \int_{S_n=\pi^{-1}S_n} M(\alpha_n^{-1}\mathcal{X})\mathrm{d}(\pi\mathcal{X})\\ &= |\pi|_\nu^n \int_{S_n} M(\alpha_n^{-1}\mathcal{X})\mathrm{d}\mathcal{X}\\ &= |\pi|_\nu^n \kappa_n(\alpha_n).\end{aligned}$$

□

**Remark:**

Considering this lemma we see that we may now simply concentrate on the $\alpha_n \in T$ such that the exponents $X_i \geq 0$ for all $1 \leq i \leq n$.

**Theorem 3.1.4** *On the torus $T \subset GL_2(k_\nu)$ the function $\kappa_2$ satisfies,*

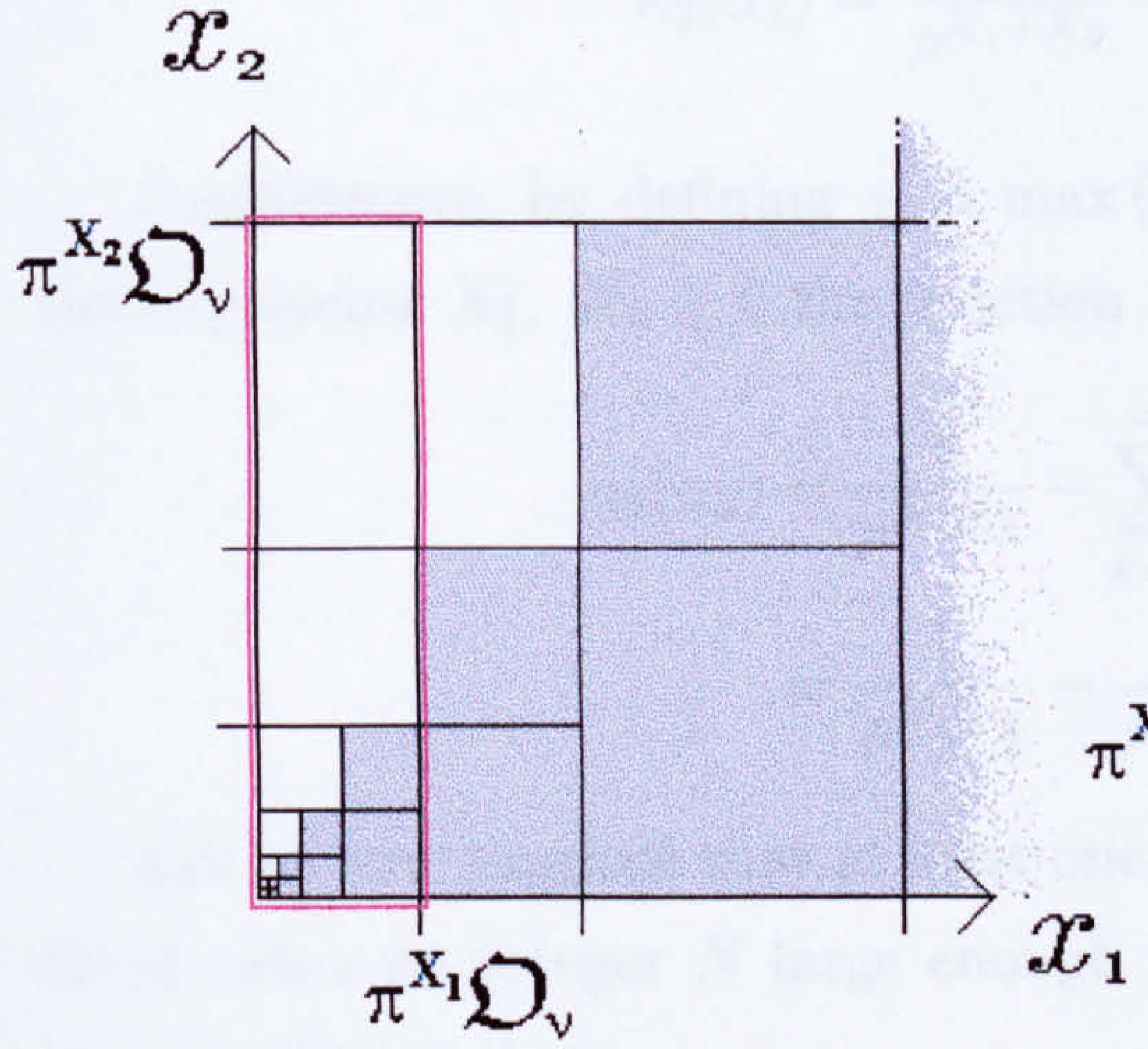
$$\begin{aligned}\kappa_2\left(\begin{pmatrix}\pi^{X_i}a_i & 0\\ 0 & \pi^{X_j}a_j\end{pmatrix}\right) &= \frac{1}{\rho^{X_i+\max(X_i,X_j)}} - \frac{(\rho-1)}{\rho^{2\max(X_i,X_j)}(\rho^2-1)}\\ &=: \kappa_2(X_i : X_j),\end{aligned}$$

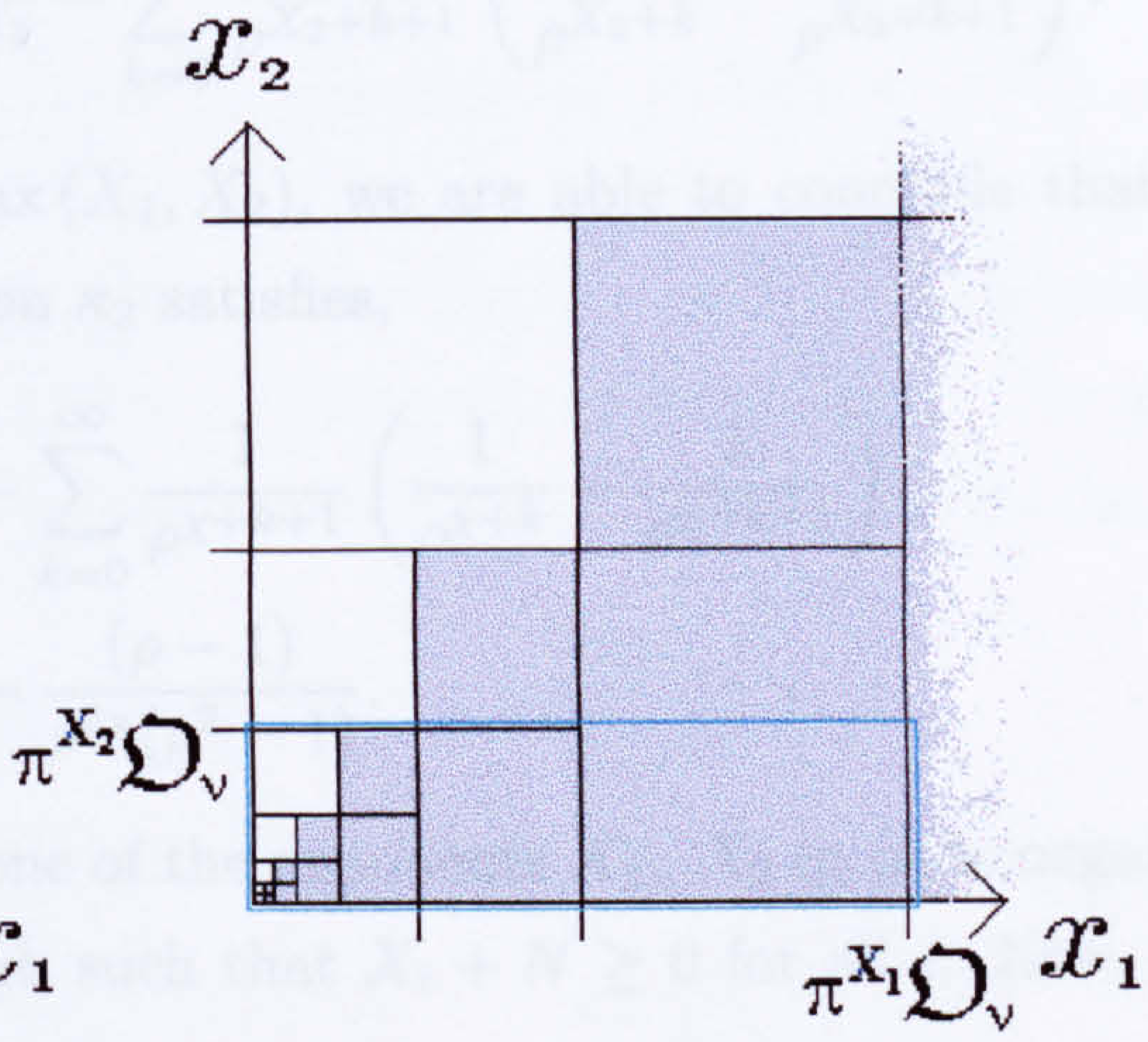
*where the notation $\kappa_2(X_i : X_j)$ is defined for the later sections.*

**PROOF OF THEOREM:**

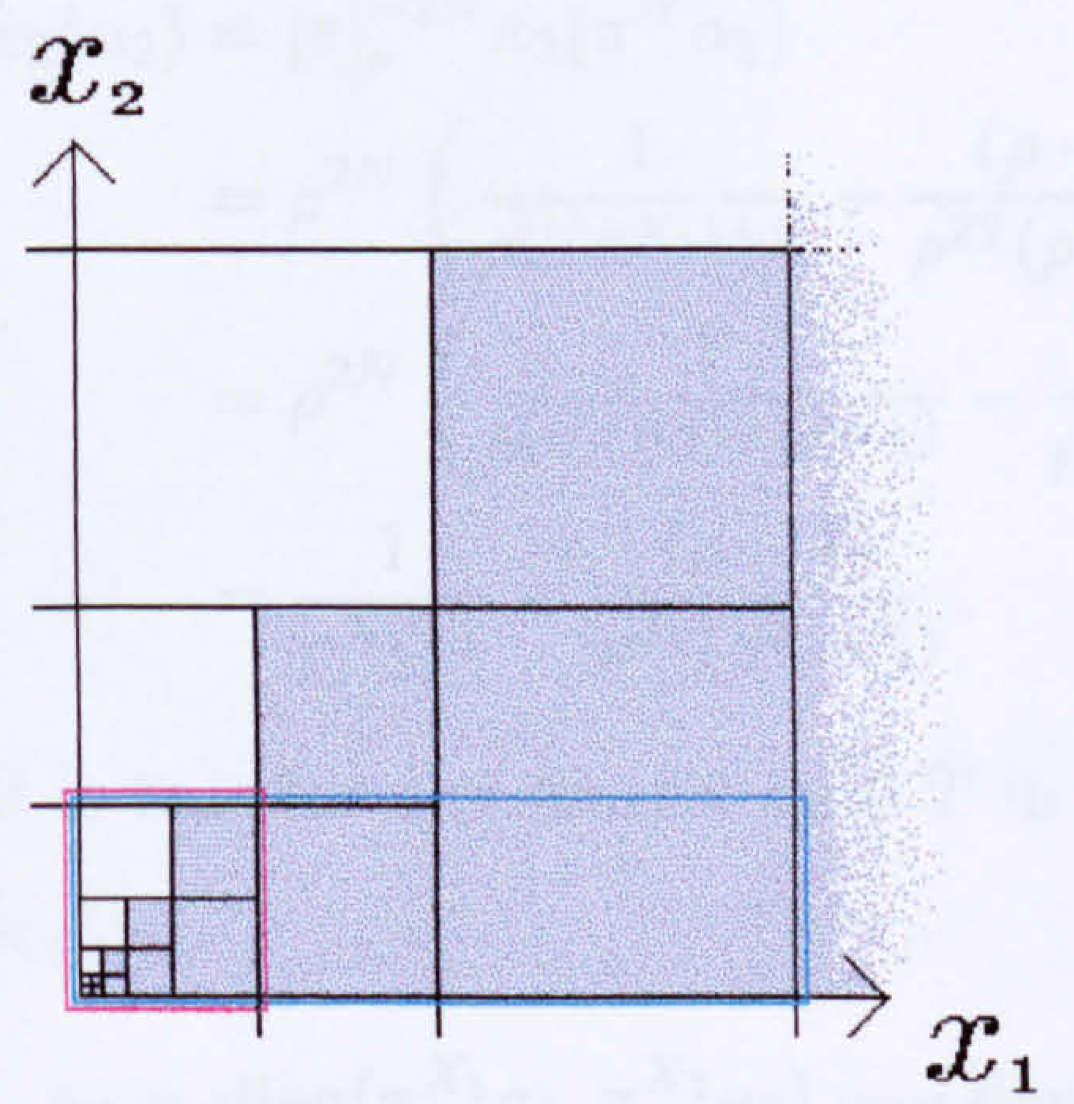
In order to prove this theorem we note that we are still able to represent the value of the function $\kappa_2(\alpha_2)$ graphically. However, in order to do so, we must appreciate that there are many cases to consider depending on the entries of $\alpha_2$.

To begin we shall assume that both the exponents $X_1$ and $X_2$ are positive. Then we have either $X_1 \geq X_2 \geq 0$ or $X_2 \geq X_1 \geq 0$. These I have represented by the shaded area enclosed within the red box and the green box respectively.




Using what we have learnt from the previous section on the identity we shall once again want to calculate these as the area of some rectangle minus the area of some unshaded region. Since we are interested in a single formula, we look at both of these areas on the same diagram and then re-draw the red box. These areas may now be calculated as either the 'new' red box or the green box minus a similar unshaded region.



We calculate these areas to be,

$$X_1 \geq X_2 \geq 0: \qquad \kappa_2(\alpha_2) = \frac{1}{\rho^{2X_1}} - \sum_{k=0}^{\infty} \frac{1}{\rho^{X_1+k+1}} \left( \frac{1}{\rho^{X_1+k}} - \frac{1}{\rho^{X_1+k+1}} \right)$$

$$X_2 \geq X_1 \geq 0: \qquad \kappa_2(\alpha_2) = \frac{1}{\rho^{X_1+X_2}} - \sum_{k=0}^{\infty} \frac{1}{\rho^{X_2+k+1}} \left( \frac{1}{\rho^{X_2+k}} - \frac{1}{\rho^{X_2+k+1}} \right).$$

Furthermore, by defining $\chi = \max(X_1, X_2)$, we are able to conclude that whenever the exponents $X_1,\ X_2 \geq 0$ the function $\kappa_2$ satisfies,

$$\begin{aligned} \kappa_2(\alpha_2) &= \frac{1}{\rho^{X_1+\chi}} - \sum_{k=0}^{\infty} \frac{1}{\rho^{\chi+k+1}} \left( \frac{1}{\rho^{\chi+k}} - \frac{1}{\rho^{\chi+k+1}} \right) \\ &= \frac{1}{\rho^{X_1+\chi}} - \frac{(\rho-1)}{\rho^{2\chi}(\rho^2-1)}. \end{aligned}$$

Let us now suppose that at least one of the exponents $X_1,\ X_2$ in $\alpha_2$ is negative. Then, there exists an integer $N$ large enough such that $X_i + N \geq 0$ for all $i$. Now, by Lemma 3.1.1, we know that

$$\kappa_2(\pi^N \alpha_2) = |\pi|_\nu^{2N} \kappa_2(\alpha_2).$$

Therefore, using our previous result and defining,

$$\bar{\chi} = \max(N + X_1, N + X_2) = N + \chi,$$

we have,

$$\begin{aligned} \kappa_2(\alpha_2) &= |\pi|_\nu^{-2N} \kappa_2(\pi^N \alpha_2) \\ &= \rho^{2N} \left( \frac{1}{\rho^{(N+X_1)+\bar{\chi}}} - \frac{(\rho-1)}{\rho^{2\bar{\chi}}(\rho^2-1)} \right) \\ &= \rho^{2N} \left( \frac{1}{\rho^{(N+X_1)+(N+\chi)}} - \frac{(\rho-1)}{\rho^{2(N+\chi)}(\rho^2-1)} \right) \\ &= \frac{1}{\rho^{X_1+\chi}} - \frac{(\rho-1)}{\rho^{2\chi}(\rho^2-1)}. \end{aligned}$$

Therefore our result is true for any matrix $\alpha_2 \in T$ in $\mathrm{GL}_2(k_\nu)$.

By substituting,

$$\alpha_2 = \mathrm{diag}(\pi^{X_1} a_1, \pi^{X_2} a_2) \longmapsto \mathrm{diag}(\pi^{X_i} a_i, \pi^{X_j} a_j),$$

our original result quickly follows. $\square$

**Theorem 3.1.5** *On the torus $T \subset GL_3(k_\nu)$ the function $\kappa_3$ satisfies,*

$$\kappa_3(\alpha_3) = \frac{1}{\rho^{2\chi_1+\chi}} - \frac{(\rho-1)}{\rho^{3\chi}(\rho^3-1)} + \frac{1}{\rho^{\chi_1}}(\kappa_2(\chi_2 : X_3) - \kappa_2(\chi_1 : X_3))$$
$$=: \kappa_3(\chi_1,\, \chi_2 : X_3),$$

*where* $\chi_1 = \max(X_1, X_2)$, $\chi_2 = \min(X_1, X_2)$ *and* $\chi = \max(X_1, X_2, X_3)$ *and the notation* $\kappa_3(\chi_1,\, \chi_2 : X_3)$ *is defined for later sections.*

**PROOF OF THEOREM:**

We are now looking for the volume of the intersection of $(\pi^{X_1}\mathfrak{O}_\nu \times \pi^{X_2}\mathfrak{O}_\nu \times \pi^{X_3}\mathfrak{O}_\nu)$ with the set $S_3$.

By considering the result of Lemma 3.1.1 we see that it shall be sufficient for us to prove this theorem for each diagonal matrix $\alpha_3$ where the exponents $X_1$, $X_2$ and $X_3$ are all positive. If we also notice that $S_3$ is symmetric in the variables $x_1$ and $x_2$ then by using the notation,

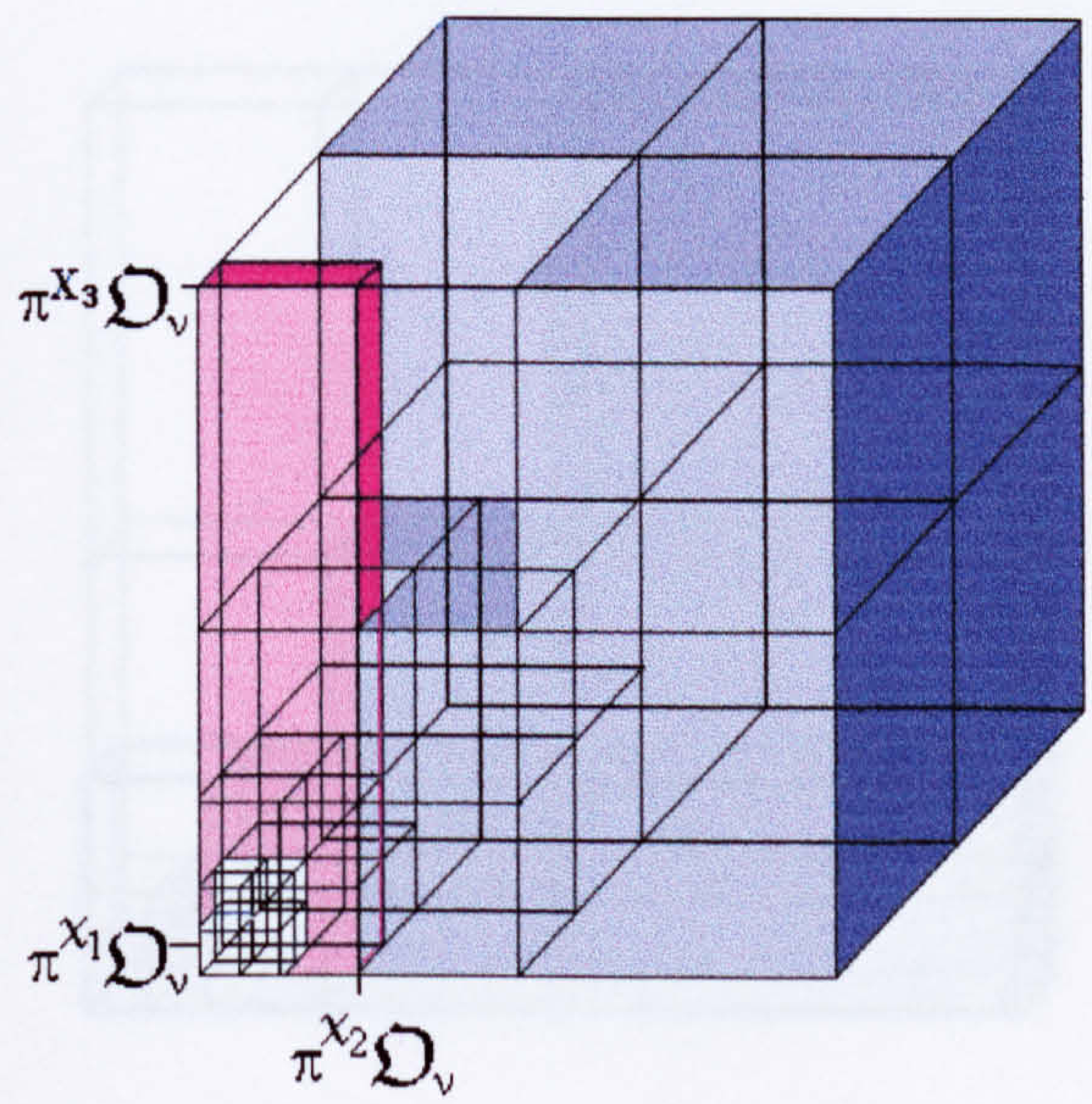
$$\chi_1 = \max(X_1, X_2) \quad \text{and} \quad \chi_2 = \min(X_1, X_2),$$

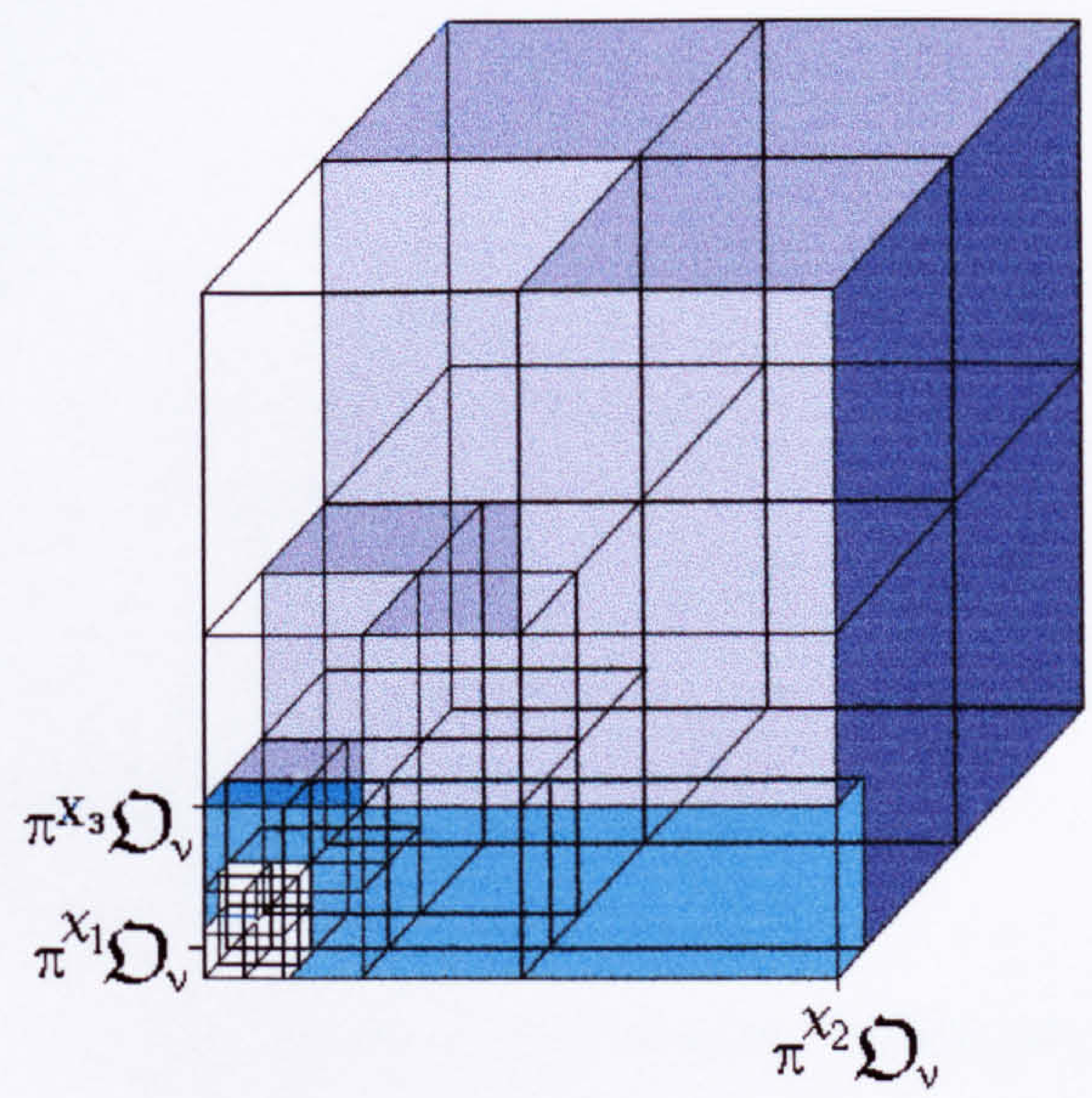
we shall have only three possible cases to consider. These are,

$$\chi_1 \geq \chi_2 \geq X_3, \qquad \chi_1 \geq X_3 \geq \chi_2, \qquad X_3 \geq \chi_1 \geq \chi_2.$$

Since we are in three dimensions we shall once again be able to represent the value of the function $\kappa_3$ graphically.

**• $\chi_1 \geq X_3$:**

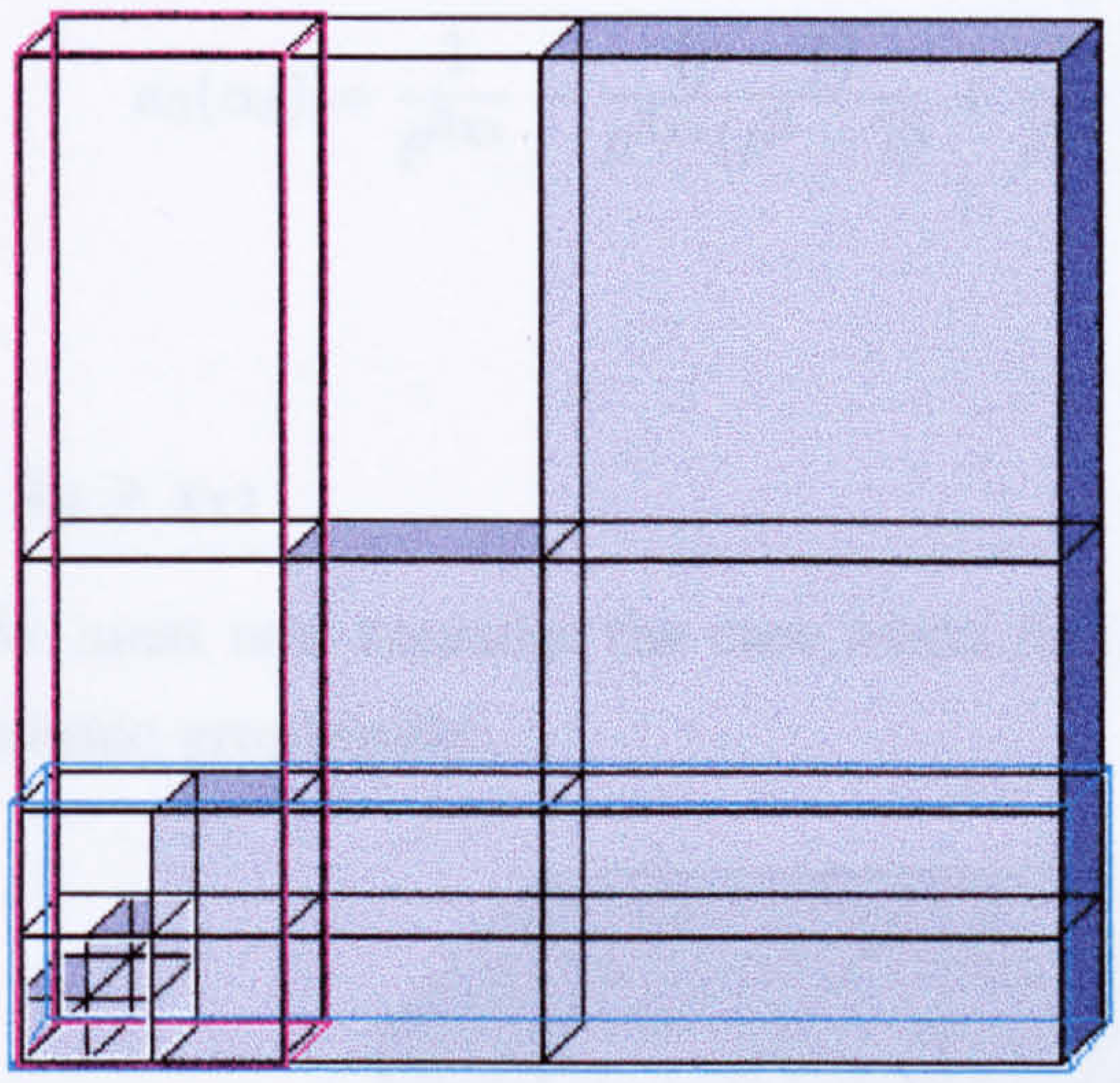


Immediately we notice that the volumes represented in yellow are identical. Therefore we split this up as the volume in yellow $\mathcal{V}_1$ and the volume which remains $\mathcal{V}_2$. For the volume $\mathcal{V}_1$ we calculate,

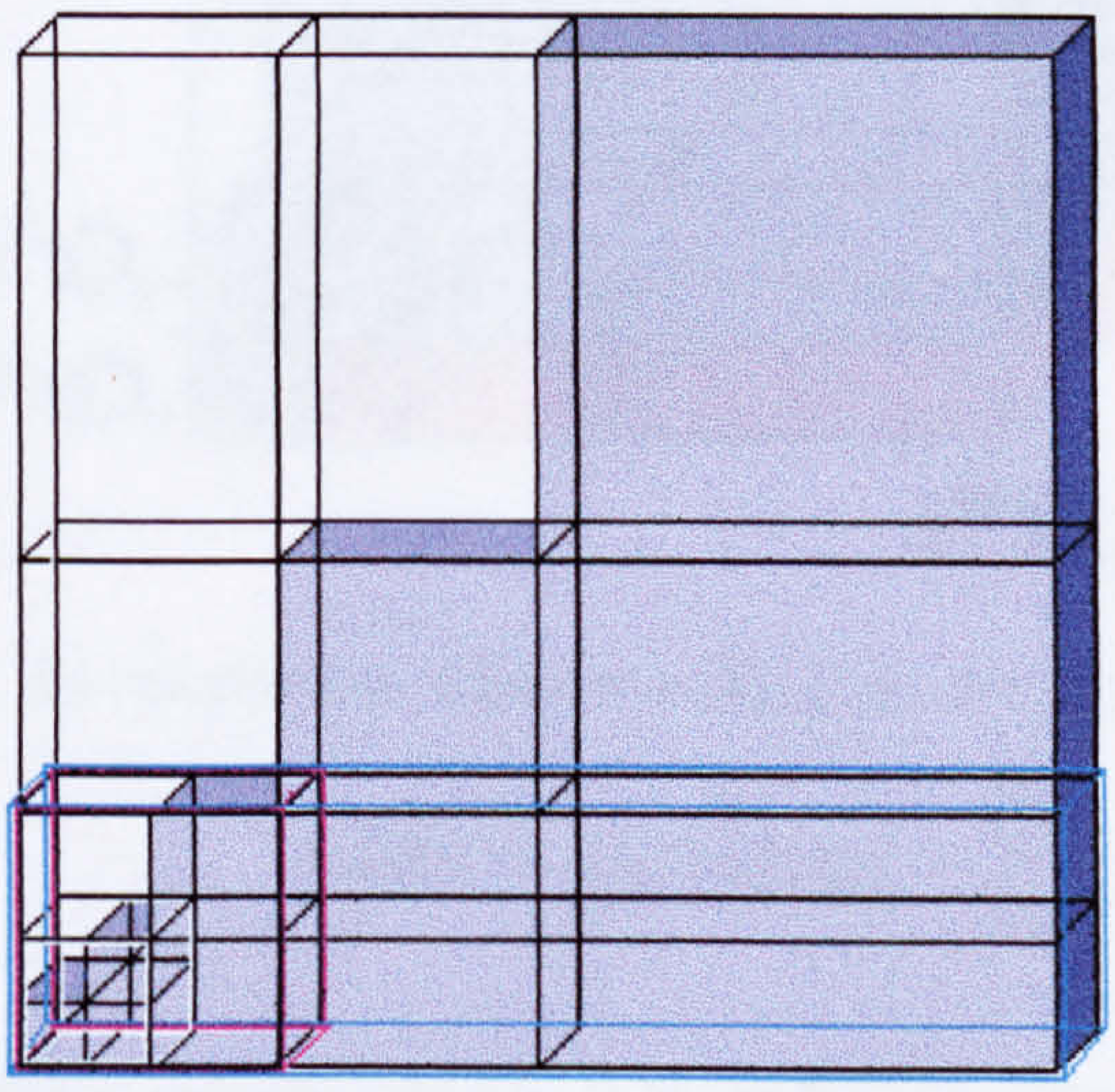
$$\mathcal{V}_1 = \frac{1}{\rho^{3\chi_1}} - \sum_{k=0}^{\infty} \frac{1}{\rho^{2(\chi_1+k+1)}} \left( \frac{1}{\rho^{\chi_1+k}} - \frac{1}{\rho^{\chi_1+k+1}} \right) = \frac{1}{\rho^{3\chi_1}} - \frac{(\rho-1)}{\rho^{3\chi_1}(\rho^3-1)}.$$

When considering the volume $\mathcal{V}_2$ which remains we see that the distance, $1/\rho^{\chi_1}$, in the $\chi_1$-dimension is constant. So we may, almost, consider this as a two dimensional problem.



That is, we are looking for the volume of the shaded region within the red and green boxes, minus the volume corresponding to the yellow box, which has already been counted.

Looking back to the proof of Theorem 3.1.4, on page 49, we realize that we may again re-draw the red box,



Using our work from the previous theorem we are then able to write the volume $\mathcal{V}_2$ as,

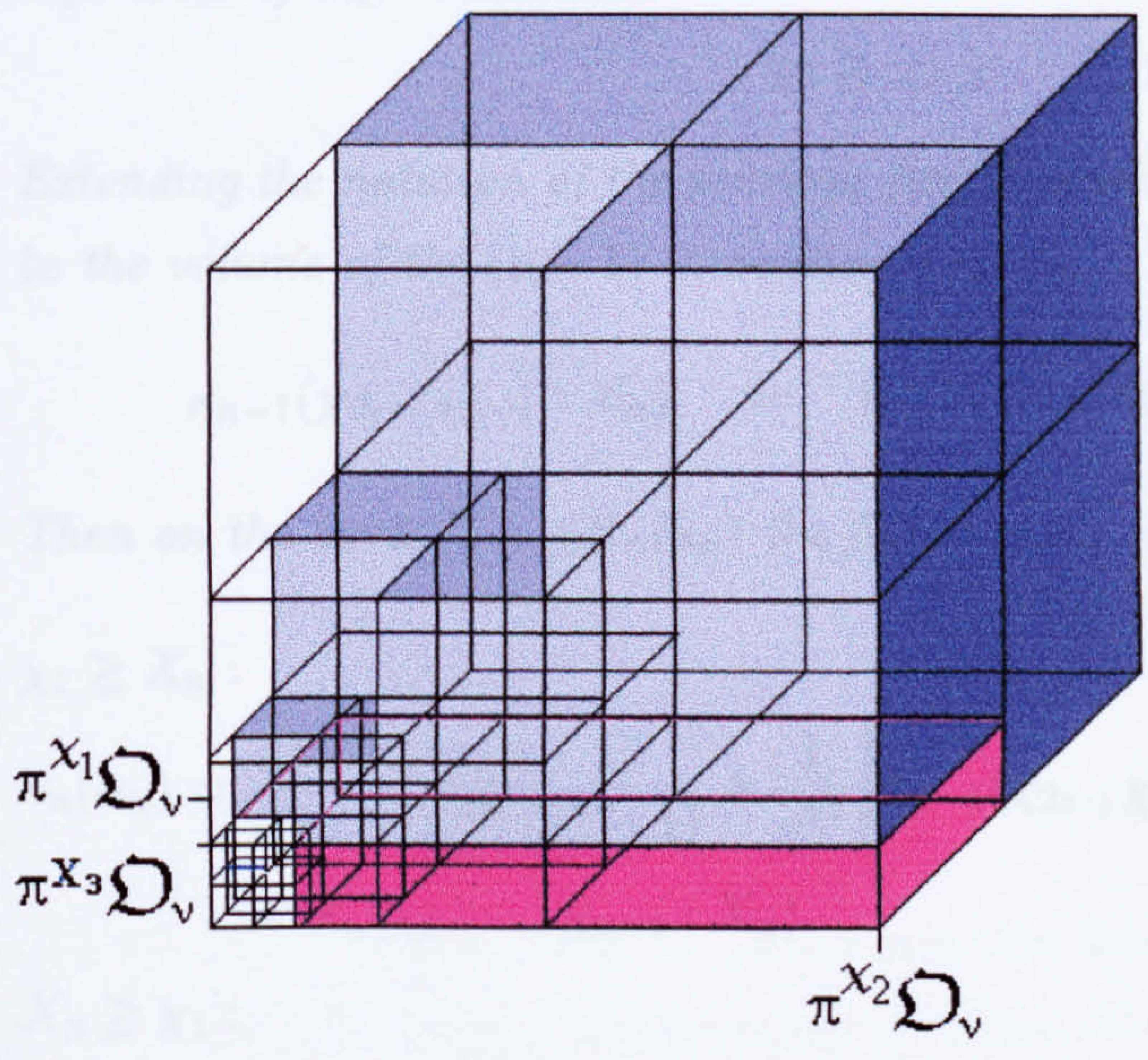
$$\begin{aligned}\mathcal{V}_2 =& \frac{1}{\rho^{\chi_1}}\Bigg[\bigg(\frac{1}{\rho^{\chi_2+\max(\chi_2,X_3)}} - \sum_{k=0}^{\infty}\frac{1}{\rho^{(\max(\chi_2,X_3)+k+1)}}\Big(\frac{1}{\rho^{\max(\chi_2,X_3)+k}} - \frac{1}{\rho^{\max(\chi_2,X_3)+k+1}}\Big)\bigg)\\ &-\bigg(\frac{1}{\rho^{2\chi_1}} - \sum_{k=0}^{\infty}\frac{1}{\rho^{(\chi_1+k+1)}}\Big(\frac{1}{\rho^{\chi_1+k}} - \frac{1}{\rho^{\chi_1+k+1}}\Big)\bigg)\Bigg].\end{aligned}$$

Having calculated the volumes $\mathcal{V}_1$ and $\mathcal{V}_2$ we may conclude that, whenever $\chi_1 \geq X_3$, the function $\kappa_3$ satisfies,

$$\kappa_3(\alpha_3) = \frac{1}{\rho^{3\chi_1}} - \frac{(\rho-1)}{\rho^{3\chi_1}(\rho^3-1)} + \frac{1}{\rho^{\chi_1}}\Big[\kappa_2(\chi_2 : X_3) - \Big(\frac{1}{\rho^{2\chi_1}} - \frac{(\rho-1)}{\rho^{2\chi_1}(\rho^2-1)}\Big)\Big].$$

- **$X_3 \geq \chi_1$:**

We must now consider the case when $X_3 \geq \chi_1$. Once again we are able to represent this volume graphically:




To calculate this we take the volume of the entire purple box and then subtract the unshaded volume from the yellow box.

In conclusion, whenever $X_3 \geq \chi_1$, we have

$$\kappa_3(\alpha_3) = \frac{1}{\rho^{(X_1+X_2+X_3)}} - \sum_{k=0}^{\infty}\frac{1}{\rho^{2(X_3+k+1)}}\Big(\frac{1}{\rho^{X_3+k}} - \frac{1}{\rho^{X_3+k+1}}\Big).$$

Having calculated $\kappa_3(\alpha_3)$ in each of the cases $\chi_1 \geq X_3$ and $X_3 \geq \chi_1$ it simply remains to express our results in a single equation.

Remembering the definition $\chi = \max(X_1, X_2, X_3)$ we may write,

$$\begin{aligned}
\kappa_3(\alpha_3) &= \frac{1}{\rho^{2\chi_1+\chi}} - \sum_{k=0}^{\infty} \frac{1}{\rho^{2(\chi+k+1)}} \left( \frac{1}{\rho^{\chi+k}} - \frac{1}{\rho^{\chi+k+1}} \right) \\
&\quad + \frac{1}{\rho^{\chi_1}} \left[ \left( \frac{1}{\rho^{\chi_2+\max(\chi_2,X_3)}} - \sum_{k=0}^{\infty} \frac{1}{\rho^{(\max(\chi_2,X_3)+k+1)}} \left( \frac{1}{\rho^{\max(\chi_2,X_3)+k}} - \frac{1}{\rho^{\max(\chi_2,X_3)+k+1}} \right) \right) \right. \\
&\qquad \left. - \left( \frac{1}{\rho^{\chi_1+\max(\chi_1,X_3)}} - \sum_{k=0}^{\infty} \frac{1}{\rho^{(\max(\chi_1,X_3)+k+1)}} \left( \frac{1}{\rho^{\max(\chi_1,X_3)+k}} - \frac{1}{\rho^{\max(\chi_1,X_3)+k+1}} \right) \right) \right] \\
&= \frac{1}{\rho^{2\chi_1+\chi}} - \frac{(\rho-1)}{\rho^{3\chi}(\rho^3-1)} + \frac{1}{\rho^{\chi_1}} (\kappa_2(\chi_2 : X_3) - \kappa_2(\chi_1 : X_3)),
\end{aligned}$$

which completes the proof of our theorem. □

**Theorem 3.1.6** *Let $\chi_1, \chi_2, \dots, \chi_{n-1}$ be any re-ordering of $X_1, X_2, \dots, X_{n-1}$, the exponents of $\alpha_n$, such that* [1]

$$\chi_1 \geq \chi_2 \geq \dots \geq \chi_{n-1}.$$

*Extending the notation of the previous few theorems let us define $\kappa_{n-1}(\chi_2, ., \chi_{n-1} : X_n)$ to be the volume of the $(n-1)$-dimensional space,*

$$\kappa_{n-1}(\chi_2, ., \chi_{n-1} : X_n) \quad := \quad |S_{n-1} \cap (\pi^{\chi_2}\mathfrak{O}_\nu \times \dots \times \pi^{\chi_{n-1}}\mathfrak{O}_\nu \times \pi^{X_n}\mathfrak{O}_\nu)|.$$

*Then on the torus $T \subset GL_n(k_\nu)$ the function $\kappa_n$ satisfies,*

$\chi_1 \geq X_n$ :

$$\begin{aligned}
\kappa_n(\alpha_n) &= \frac{1}{\rho^{n\chi_1}} - \frac{(\rho-1)}{\rho^{n\chi_1}(\rho^n-1)} + \frac{1}{\rho^{\chi_1}} \left[ \kappa_{n-1}(\chi_2, ., \chi_{n-1} : X_n) - \left( \frac{1}{\rho^{(n-1)\chi_1}} - \frac{(\rho-1)}{\rho^{(n-1)\chi_1}(\rho^{n-1}-1)} \right) \right] \\
&=: \kappa_n(\chi_1, \dots, \chi_{n-1} : X_n)
\end{aligned}$$

$X_n \geq \chi_1$ :

$$\begin{aligned}
\kappa_n(\alpha_n) &= \frac{1}{\rho^{(X_1+\dots+X_n)}} - \frac{(\rho-1)}{\rho^{nX_n}(\rho^n-1)} \\
&=: \kappa_n(\chi_1, \dots, \chi_{n-1} : X_n).
\end{aligned}$$

Although the previous theorems have allowed us to state our results in a single formula, for the purpose of the work to follow it is more convenient for us to leave this theorem as two cases.

[1] note that this is consistent with our definitions of $\chi_1$ and $\chi_2$ in the previous theorem.

**PROOF OF THEOREM:**

By using the result,

$$\kappa_n(\pi^N \alpha_n) = |\pi|_\nu^{nN} \kappa_n(\alpha_n),$$

given in Lemma 3.1.1 we see that in order to prove this theorem we need only consider diagonal matrices $\alpha_n$ where each of the exponents $X_i \geq 0,\ 1 \leq i \leq n$.

• $\chi_1 \geq X_n$:

In order to calculate the function $\kappa_n(\alpha_n)$ when $\chi_1 \geq X_n$ we carefully consider the proof of Theorem 3.1.5. On page 51 we found the value of $\kappa_3(\alpha_3)$ by splitting the calculation into two parts and considering the volumes $\mathcal{V}_1$ and $\mathcal{V}_2$ separately. We shall now repeat this procedure for any dimension $n$.

**The volume $\mathcal{V}_1$**

Firstly we shall consider the volume $\mathcal{V}_1$. If we had been able to represent this graphically $\mathcal{V}_1$ would have been the volume of the shaded region within the $n$-dimensional cube $(\pi^{\chi_1}\mathfrak{O}_\nu)^n$. That is,

$$\begin{aligned}\mathcal{V}_1 = |S_n \cap (\pi^{\chi_1}\mathfrak{O}_\nu)^n| &= \frac{1}{\rho^{n\chi_1}} - \sum_{k=0}^{\infty} \frac{1}{\rho^{(n-1)(\chi_1+k+1)}} \Big( \frac{1}{\rho^{\chi_1+k}} - \frac{1}{\rho^{\chi_1+k+1}} \Big) \\ &= \frac{1}{\rho^{n\chi_1}} - \frac{(\rho-1)}{\rho^{n\chi_1}(\rho^n-1)}.\end{aligned}$$

**The volume $\mathcal{V}_2$**

Had we been able to draw all of the possible sets with $\chi_1 \geq X_n$, we would have noticed that in each case the distance in the $\chi_1$-dimension was constant (ie. $1/\rho^{\chi_1}$). This allows us to project onto this dimension and consider the $(n-1)$-dimensional volume of the set which remains. Using the notation we defined earlier this volume is given by,

$$\kappa_{n-1}(\chi_2, .., \chi_{n-1} : X_n) = |S_{n-1} \cap (\pi^{\chi_2}\mathfrak{O}_\nu \times \ldots \times \pi^{\chi_{n-1}}\mathfrak{O}_\nu \times \pi^{X_n}\mathfrak{O}_\nu)|.$$

Finally, to conclude the calculation of $\mathcal{V}_2$ we must subtract the volume of the cube $(\pi^{\chi_1}\mathfrak{O}_\nu)^{(n-1)}$ intersected with $S_{n-1}$ as this has already been counted in $\mathcal{V}_1$.

Therefore we eventually find that the required volume $\mathcal{V}_2$ satisfies,

$$\begin{aligned}
\mathcal{V}_2 &= \frac{1}{\rho^{\chi_1}}\Big[|S_{n-1}\cap(\pi^{\chi_2}\mathfrak{O}_\nu\times\ldots\times\pi^{X_n}\mathfrak{O}_\nu)| - |S_{n-1}\cap(\pi^{\chi_1}\mathfrak{O}_\nu)^{n-1}|\Big] \\
&= \frac{1}{\rho^{\chi_1}}\Big[\kappa_{n-1}(\chi_2,.,\chi_{n-1}:X_n) - \Big(\frac{1}{\rho^{(n-1)\chi_1}} - \sum_{k=0}^{\infty}\frac{1}{\rho^{(n-2)(\chi_1+k+1)}}\Big(\frac{1}{\rho^{\chi_1+k}} - \frac{1}{\rho^{\chi_1+k+1}}\Big)\Big)\Big] \\
&= \frac{1}{\rho^{\chi_1}}\Big[\kappa_{n-1}(\chi_2,.,\chi_{n-1}:X_n) - \Big(\frac{1}{\rho^{(n-1)\chi_1}} - \frac{(\rho-1)}{\rho^{(n-1)\chi_1}(\rho^{n-1}-1)}\Big)\Big].
\end{aligned}$$

Finally, by substituting our results for $\mathcal{V}_1$ and $\mathcal{V}_2$ we do indeed find that,

$$\kappa_n(\alpha_n) = \frac{1}{\rho^{n\chi_1}} - \frac{(\rho-1)}{\rho^{n\chi_1}(\rho^n-1)} + \frac{1}{\rho^{\chi_1}}\Big[\kappa_{n-1}(\chi_2,.,\chi_{n-1}:X_n) - \Big(\frac{1}{\rho^{(n-1)\chi_1}} - \frac{(\rho-1)}{\rho^{(n-1)\chi_1}(\rho^{n-1}-1)}\Big)\Big].$$

• $X_n \geq \chi_1$:

Let us once again consider the proof of Theorem 3.1.5. There we calculated $\kappa_3(\alpha_3)$ as the volume of the $n$-dimensional box $(\pi^{X_1}\mathfrak{O}_\nu\times\ldots\times\pi^{X_n}\mathfrak{O}_\nu)$ minus the unshaded volume within the $n$-dimensional cube $(\pi^{X_n}\mathfrak{O}_\nu)^n$. Repeating this procedure we find that, in any dimension $n$, when $X_n$ is the greatest of the exponents the function $\kappa_n(\alpha_n)$ satisfies,

$$\begin{aligned}
\kappa_n(\alpha_n) &= \frac{1}{\rho^{(X_1+\ldots+X_n)}} - \sum_{k=0}^{\infty}\frac{1}{\rho^{(n-1)(X_n+k+1)}}\Big(\frac{1}{\rho^{X_n+k}} - \frac{1}{\rho^{X_n+k+1}}\Big) \\
&= \frac{1}{\rho^{(X_1+\ldots+X_n)}} - \frac{(\rho-1)}{\rho^{nX_n}(p^n-1)}.
\end{aligned}$$

□

**Remark:**

It is worth pointing out here that, as with the previous section when $\chi_1 \geq X_n$, we could have calculated $\kappa_n(\alpha_n)$ by splitting into volumes $\mathcal{V}_1$ and $\mathcal{V}_2$. If we had done so for $X_n \geq \chi_1$ we would have found,

$$\begin{aligned}
\kappa_n(\alpha_n) &= \mathcal{V}_1 + \mathcal{V}_2 \\
&= |S_n\cap(\pi^{X_n}\mathfrak{O}_\nu)^n| + \frac{1}{\rho^{X_n}}\Big[|\mathfrak{O}_\nu^{n-1}\cap(\pi^{\chi_1}\mathfrak{O}_\nu\times.\times\pi^{\chi_{n-1}}\mathfrak{O}_\nu)| - |\mathfrak{O}_\nu^{n-1}\cap(\pi^{X_n}\mathfrak{O}_\nu)^{n-1}|\Big].
\end{aligned}$$

with the similarities between the two cases clear to see.

It should now be clear why it is difficult to summarise both results by a single formula. Since $\mathcal{V}_1$ is easily represented in both cases using $\chi := \max\{\chi_1, X_n\}$ the problem clearly lies with $\mathcal{V}_2$.

In the case that $\chi_1 \geq X_n$, projecting onto the $\chi_1$-dimension leaves us searching for the volume of a subset of $S_{n-1}$. However, when $X_n \geq \chi_1$, projecting onto the $X_n$-dimension leaves a subset of $\mathfrak{O}_\nu^{n-1}$. This is most clearly illustrated by the projection of the space $S_n$ onto each pair of dimensions as on page 47. Unfortunately these volumes are not easily related in mathematical formulae.

## 3.2 The function $\jmath_n$

Having calculated the values of $\kappa_n$ we are now in a position to consider the function

$$\jmath_n(\alpha_n) = (-1)^{\frac{1}{2^{2r}}\left(J_n(\alpha_n) - \left(1 - |\pi^{X_1+\ldots+X_{n-1}}a_1\ldots a_{n-1}|_\nu\right)\right)}.$$

Before we begin this section we shall first consider a small lemma which shall be important in calculations throughout this thesis.

**Lemma 3.2.1** *Let $\rho$ be as previously defined, then we know that $\rho \equiv 1 \pmod m$, where $m = 2^r$ is the number of roots of unity in $\mu_m$. So, for any integers $\theta, k \in \mathbb{Z}$ we have,*

$$\rho^\theta \equiv 1/\rho^\theta, \quad \rho^{2k\theta} \equiv 1 \quad \textit{and} \quad \rho^{(2k+1)\theta} \equiv \rho^\theta \pmod{2^{r+1}}.$$

*We also find,*

- if $\rho \equiv 1 \pmod{2^{r+1}}$ :

$$\begin{aligned}(\rho^{2k(\theta-1)} + \rho^{2k(\theta-2)} + \ldots + \rho^{2k} + 1) &\equiv \theta\\ (\rho^{(2k+1)(\theta-1)} + \rho^{(2k+1)(\theta-2)} + \ldots + \rho^{2k+1} + 1) &\equiv \theta.\end{aligned}$$

- if $\rho \equiv 2^r + 1 \pmod{2^{r+1}}$ :

$$\begin{aligned}(\rho^{2k(\theta-1)} + \rho^{2k(\theta-2)} + \ldots + \rho^{2k} + 1) &\equiv \theta\\ (\rho^{(2k+1)(\theta-1)} + \rho^{(2k+1)(\theta-2)} + \ldots + \rho^{2k+1} + 1) &\equiv \begin{cases}(2^{r-1}+1)\theta & \textit{if } \theta \textit{ is even}\\ (2^{r-1}+1)\theta - 2^{r-1} & \textit{if } \theta \textit{ is odd.}\end{cases}\end{aligned}$$

**PROOF:**

The proof of these statements relies on the fact that $\rho \equiv 1 \pmod{2^r}$. Then by simply counting the terms in each expression our result quickly follows. □

### 3.2.1 $\jmath_2$ on $T \subset \mathrm{GL}_2(k_\nu)$

**Theorem 3.2.1** *The function $\jmath_2$ on the torus $T \subset GL_2(k_\nu)$ satisfies,*

$$\begin{aligned}\jmath_2(\alpha_2) &= (-1)^{\frac{(\rho-1)}{m}\frac{\max(X_1,X_2)(\max(X_1,X_2)+2X_1+1)}{2}} \\ &= (-1)^{\frac{(\rho-1)}{2^r}X_1\max(X_1,X_2)}(-1)^{\frac{(\rho-1)}{2^r}\frac{\max(X_1,X_2)(\max(X_1,X_2)+1)}{2}} \\ &=: \jmath_2(X_1:X_2),\end{aligned}$$

*where the notation $\jmath_2(X_1 : X_2)$ is defined for the later theorems.*

**PROOF OF THEOREM:**

As we have seen previously the function $\jmath_2$ is defined by,

$$\begin{aligned}\jmath_2(\alpha_2) &= (-1)^{\frac{1}{m^2}\left(J_2(\alpha_2)-(1-\frac{1}{\rho^{X_1}})\right)} \\ &= (-1)^{\frac{1}{2^{2r}}\left(\kappa_2(I_2)-\kappa_2(\alpha_2)-(1-\frac{1}{\rho^{X_1}})\right)}.\end{aligned}$$

By defining,

$$\nabla := \kappa_2(I_2)-\kappa_2(\alpha_2)-(1-\frac{1}{\rho^{X_1}}),$$

and using the results of the previous section we find,

$$\begin{aligned}\frac{1}{m^2}\nabla &= \frac{1}{m^2}\left[\left(1-\frac{(\rho-1)}{(\rho^2-1)}\right)-\left(\frac{1}{\rho^{X_1+\chi}}-\frac{(\rho-1)}{\rho^{2\chi}(\rho^2-1)}\right)-\left(1-\frac{1}{\rho^{X_1}}\right)\right] \\ &= \frac{1}{m^2}\left[\left(1-\frac{1}{\rho^{X_1+\chi}}\right)-\frac{(\rho-1)}{(\rho^2-1)}\left(1-\frac{1}{\rho^{2\chi}}\right)-\left(1-\frac{1}{\rho^{X_1}}\right)\right] \\ &= \frac{1}{m^2}\left[\frac{1}{\rho^{X_1+\chi}}(\rho^{X_1+\chi}-1)-\frac{(\rho-1)}{\rho^{2\chi}(\rho^2-1)}(\rho^{2\chi}-1)-\frac{1}{\rho^{X_1}}(\rho^{X_1}-1)\right] \\ &= \frac{(\rho-1)}{2^r}\cdot\frac{1}{2^r}\left[\frac{1}{\rho^{X_1+\chi}}(\rho^{(X_1+\chi)-1}+.+1)-\frac{1}{\rho^{2\chi}}(\rho^{2(\chi-1)}+.+1)-\frac{1}{\rho^{X_1}}(\rho^{X_1-1}+.+1)\right],\end{aligned}$$

where once again we have $\chi = \max(X_1, X_2)$.

Let us now define the expression,

$$\mathfrak{T}_1 := \left[\frac{1}{\rho^{X_1+\chi}}(\rho^{(X_1+\chi)-1}+.+1)-\frac{1}{\rho^{2\chi}}(\rho^{2(\chi-1)}+.+1)-\frac{1}{\rho^{X_1}}(\rho^{X_1-1}+.+1)\right].$$

In order to calculate the function $\jmath_n$ it simply remains to consider $\mathfrak{T}_1$ modulo $2^{r+1}$. Since this will depend on the value of $\rho \pmod{2^{r+1}}$ we shall consider each case separately.

**Suppose** $\rho \equiv 1 \pmod{2^{r+1}}$:

Since we have $(\rho - 1) \equiv 0 \pmod{2^{r+1}}$, by Lemma 3.2.1, we find

$$\mathfrak{T}_1 \equiv X_1 + \chi - \chi - X_1 \equiv 0 \pmod{2^{r+1}}.$$

In conclusion we have found that $\jmath_2(\alpha_2)$ is trivial over $\mu_{2^r}$ whenever $\rho \equiv 1 \pmod{2^{r+1}}$. Therefore we may indeed write,

$$\jmath_2(\alpha_2) = (-1)^{\frac{(\rho-1)}{2^r}\frac{\chi(\chi+2X_1+1)}{2}} = 1.$$

**Suppose** $\rho \equiv 2^r + 1 \pmod{2^{r+1}}$:

We now calculate $\mathfrak{T}_1 \pmod{2^{r+1}}$ by going through the four possible cases of $X_1$ and $\chi$ being even or odd. Using Lemma 3.2.1 and calculating modulo $2^{r+1}$ we find,

| $X_1$ | $\chi$ | | | |
|---|---|---|---|---|
| even | even: | $\equiv (2^{r-1}+1)(X_1+\chi)$ | $-\chi$ | $-(2^{r-1}+1)X_1$ |
| even | odd: | $\equiv (2^r+1)((2^{r-1}+1)(X_1+\chi) - 2^{r-1})$ | $-\chi$ | $-(2^{r-1}+1)X_1$ |
| odd | even: | $\equiv (2^r+1)((2^{r-1}+1)(X_1+\chi) - 2^{r-1})$ | $-\chi$ | $-(2^r+1)((2^{r-1}+1)X_1 - 2^{r-1})$ |
| odd | odd: | $\equiv (2^{r-1}+1)(X_1+\chi)$ | $-\chi$ | $-(2^r+1)((2^{r-1}+1)X_1 - 2^{r-1})$. |

Tidying up the expressions we have,

| $X_1$ | $\chi$ | |
|---|---|---|
| even | even: | $\equiv 2^{r-1}\chi$ |
| even | odd: | $\equiv 2^r(2^{r-1}+1)X_1 + 2^{r-1}(2^r+2+1)\chi - 2^{r-1}(2^r+1)$ |
| odd | even: | $\equiv 2^{r-1}(2^r+2+1)\chi$ |
| odd | odd: | $\equiv -2^r(2^{r-1}+1)X_1 + 2^{r-1}\chi + 2^{r-1}(2^r+1)$. |

Since in the original expression we are in fact only interested in $\mathfrak{T}_1/2^r \pmod 2$ we may divide out the factor $2^{r-1}$ and then simply consider what remains modulo 4. Having done so we find,

| $X_1$ | $\chi$ | | | | | |
|---|---|---|---|---|---|---|
| even | even: | $\equiv$ | | $\chi$ | | case (i) |
| even | odd: | $\equiv (2^r+2)X_1$ | | $+(2^r+2+1)\chi$ | $-(2^r+1)$ | case (ii) |
| odd | even: | $\equiv$ | | $(2^r+2+1)\chi$ | | case (iii) |
| odd | odd: | $\equiv -(2^r+2)X_1$ | | $+\chi$ | $+(2^r+1)$ | case (iv). |

### Cases (i) and (iii)

Since we know that $(-1)^{-1} = (-1)$ the overall sign of the expression is not important. Therefore in cases (i) and (iii) we may conclude that,

$$\jmath_2(\alpha_2) = (-1)^{\frac{(\rho-1)}{2^r}\frac{\chi}{2}}.$$

However, since we also have $(-1)^{(2k+1)} = (-1)$ for each $k \in \mathbb{Z}$ and in both of these cases the expression $(\chi + 2X_1 + 1)$ is odd, we are free to express this as,

$$\jmath_2(\alpha_2) = \left[(-1)^{\frac{(\rho-1)}{2^r}\frac{\chi}{2}}\right]^{(\chi+2X_1+1)} = (-1)^{\frac{(\rho-1)}{2^r}\frac{\chi(\chi+2X_1+1)}{2}}.$$

### Cases (ii) and (iv)

Here we must also consider the value of the integer $r$, where $m = 2^r$. Re-arranging the expressions for cases (ii) and (iv) we find that modulo 4 they satisfy,

$$(2^r+2)X_1 + (2^r+2+1)\chi - (2^r+1) \equiv \begin{cases} 2X_1 - \chi - 1 \equiv (\chi + 2X_1 + 1) & r > 1 \\ \chi + 1 \equiv (\chi + 2X_1 + 1) & r = 1, \end{cases}$$

as $X_1$ is even.

$$-(2^r+2)X_1 + \chi + (2^r+1) \equiv \begin{cases} -2X_1 + \chi + 1 \equiv (\chi + 2X_1 + 1) & r > 1 \\ \chi - 1 \equiv (\chi + 2X_1 + 1) & r = 1, \end{cases}$$

as $X_1$ is odd.

These results may be summarised by,

$$\jmath_2(\alpha_2) = (-1)^{\frac{(\rho-1)}{2^r}\frac{(\chi+2X_1+1)}{2}}.$$

Finally, since in both cases $\chi$ is odd, we are able to write

$$\jmath_2(\alpha_2) = \left[(-1)^{\frac{(\rho-1)}{2^r}\frac{(\chi+2X_1+1)}{2}}\right]^{\chi} = (-1)^{\frac{(\rho-1)}{2^r}\frac{\chi(\chi+2X_1+1)}{2}}.$$

In conclusion we have shown that on the torus in $\mathrm{GL}_2(k_\nu)$ the function $\jmath_2$ satisfies,

$$\begin{aligned}\jmath_2(\alpha_2) &= (-1)^{\frac{(\rho-1)}{m}\frac{\max(X_1,X_2)(\max(X_1,X_2)+2X_1+1)}{2}} \\ &= (-1)^{\frac{(\rho-1)}{2^r}X_1\max(X_1,X_2)}(-1)^{\frac{(\rho-1)}{2^r}\frac{\max(X_1,X_2)(\max(X_1,X_2)+1)}{2}}.\end{aligned}$$

□

### 3.2.2 $\jmath_3$ on $T \subset \mathrm{GL}_3(k_\nu)$

**Theorem 3.2.2** *The function $\jmath_3$ on the torus $T \subset GL_3(k_\nu)$ satisfies,*

$$\begin{aligned}\jmath_3(\alpha_3) &= \jmath_2\left(\begin{pmatrix} \pi^{X_1}a_1 & 0 \\ 0 & \pi^{X_3}a_3 \end{pmatrix}\right)\jmath_2\left(\begin{pmatrix} \pi^{X_2}a_2 & 0 \\ 0 & \pi^{X_3}a_3 \end{pmatrix}\right) \\ &= \prod_{i=1}^{2}(-1)^{\frac{(\rho-1)}{2^r}X_i \max(X_i,X_3)}(-1)^{\frac{(\rho-1)}{2}\frac{\max(X_i,X_3)(\max(X_i,X_3)+1)}{2}}.\end{aligned}$$

**PROOF OF THEOREM:**

We have defined the function $\jmath_3$ on $T \subset \mathrm{GL}_3(k_\nu)$ by,

$$\begin{aligned}\jmath_3(\alpha_3) &= (-1)^{\frac{1}{m^2}\left(J_3(\alpha_3)-\left(1-\frac{1}{\rho^{X_1+X_2}}\right)\right)} \\ &= (-1)^{\frac{1}{2^{2r}}\left(\kappa_3(I_3)-\kappa_3(\alpha_3)-\left(1-\frac{1}{\rho^{X_1+X_2}}\right)\right)}.\end{aligned}$$

Throughout this proof we shall once again employ the definitions,

$$\chi_1 = \max(X_1, X_2), \quad \chi_2 = \min(X_1, X_2) \quad \text{and} \quad \chi = \max(X_1, X_2, X_3).$$

Having re-arranged the expression for the function $\kappa_3$ described in Theorem 3.1.5 we find that,

$$\begin{aligned}\kappa_3(\alpha_3) &= \frac{1}{\rho^{2\chi_1+\chi}} - \frac{(\rho-1)}{\rho^{3\chi}(\rho^3-1)} + \frac{1}{\rho^{\chi_1}}(\kappa_2(\chi_2 : X_3) - \kappa_2(\chi_1 : X_3)) \\ &= \frac{1}{\rho^{2\chi_1+\chi}} - \frac{(\rho-1)}{\rho^{3\chi}(\rho^3-1)} + \frac{1}{\rho^{\chi_1}}\Big((\kappa_2(I_2) - \kappa_2(\chi_1 : X_3)) - (\kappa_2(I_2) - \kappa_2(\chi_2 : X_3))\Big) \\ &= \frac{1}{\rho^{2\chi_1+\chi}} - \frac{(\rho-1)}{\rho^{3\chi}(\rho^3-1)} + \frac{1}{\rho^{\chi_1}}(J_2(\chi_1 : X_3) - J_2(\chi_2 : X_3)),\end{aligned}$$

where the notation $J_2(- : -) := \kappa_2(I_2) - \kappa_2(- : -)$.

Using this result we are able to calculate,

$$J_3(\alpha_3) - \left(1 - \frac{1}{\rho^{X_1+X_2}}\right) = \kappa_3(I_3) - \kappa_3(\alpha_3) - \left(1 - \frac{1}{\rho^{X_1+X_2}}\right)$$

$$= \left(1 - \tfrac{(\rho-1)}{(\rho^3-1)}\right) - \left(\tfrac{1}{\rho^{2\chi_1+\chi}} - \tfrac{(\rho-1)}{\rho^{3\chi}(\rho^3-1)} + \tfrac{1}{\rho^{\chi_1}}\Big(J_2(\chi_1 : X_3) - J_2(\chi_2 : X_3)\Big)\right) - \left(1 - \tfrac{1}{\rho^{X_1+X_2}}\right)$$

$$= \left[\frac{1}{\rho^{2\chi_1+\chi}}(\rho^{2\chi_1+\chi} - 1) - \frac{(\rho-1)}{\rho^{3\chi}(\rho^3-1)}(\rho^{3\chi} - 1) - \frac{1}{\rho^{X_1+X_2}}(\rho^{X_1+X_2} - 1) + \frac{1}{\rho^{\chi_1+\chi_2}}(\rho^{\chi_2-\chi_1} - 1)\right]$$
$$- \frac{1}{\rho^{\chi_1}}\left[\left(J_2(\chi_1 : X_3) - \left(1 - \frac{1}{\rho^{\chi_1}}\right)\right) - \left(J_2(\chi_2 : X_3) - \left(1 - \frac{1}{\rho^{\chi_2}}\right)\right)\right]$$

$$= (\rho-1)\left[\frac{(\rho^{(2\chi_1+\chi)-1} + . + 1)}{\rho^{2\chi_1+\chi}} - \frac{(\rho^{3(\chi-1)} + . + 1)}{\rho^{3\chi}} - \frac{(\rho^{(X_1+X_2)-1} + . + 1)}{\rho^{X_1+X_2}} + \frac{(\rho^{(\chi_2-\chi_1)-1} + . + 1)}{\rho^{\chi_1+\chi_2}}\right]$$
$$- \frac{1}{\rho^{\chi_1}}\left[\left(J_2(\chi_1 : X_3) - \left(1 - \frac{1}{\rho^{\chi_1}}\right)\right) - \left(J_2(\chi_2 : X_3) - \left(1 - \frac{1}{\rho^{\chi_2}}\right)\right)\right].$$

Therefore, having defined the expression

$$\mathfrak{T}_2 := \left[\frac{(\rho^{(2\chi_1+\chi)-1} + . + 1)}{\rho^{2\chi_1+\chi}} - \frac{(\rho^{3(\chi-1)} + . + 1)}{\rho^{3\chi}} - \frac{(\rho^{(X_1+X_2)-1} + . + 1)}{\rho^{X_1+X_2}} + \frac{(\rho^{(\chi_2-\chi_1)-1} + . + 1)}{\rho^{\chi_1+\chi_2}}\right],$$

we see that the function $\jmath_3$ satisfies,

$$\jmath_3(\alpha_3) = (-1)^{\frac{(\rho-1)}{m}\frac{\mathfrak{T}_2}{2^r}}.(-1)^{\frac{1}{m^2}\left(J_2(\chi_1:X_3)-\left(1-\frac{1}{\rho^{\chi_1}}\right)\right)}(-1)^{\frac{1}{m^2}\left(J_2(\chi_2:X_3)-\left(1-\frac{1}{\rho^{\chi_2}}\right)\right)}.$$

Concentrating on the expression $\mathfrak{T}_2$ we see that, since $\{X_1, X_2\} = \{\chi_1, \chi_2\}$, we may re-write this as

$$\mathfrak{T}_2 = \left[\frac{(\rho^{(2\chi_1+\chi)-1} + . + 1)}{\rho^{2\chi_1+\chi}} - \frac{(\rho^{3(\chi-1)} + . + 1)}{\rho^{3\chi}} - \frac{(\rho^{(\chi_1+\chi_2)-1} + . + 1)}{\rho^{\chi_1+\chi_2}} + \frac{(\rho^{(\chi_2-\chi_1)-1} + . + 1)}{\rho^{\chi_1+\chi_2}}\right]$$
$$\equiv \frac{1}{\rho^{\chi}}\left[(\rho^{(2\chi_1+\chi)-1} + \ldots + 1) - (\rho^{(\chi-1)} + \ldots + 1)\right]$$
$$- \frac{1}{\rho^{\chi_1+\chi_2}}\left[(\rho^{(\chi_1+\chi_2)-1} + \ldots + 1) - (\rho^{(\chi_2-\chi_1)-1} + \ldots + 1)\right] \pmod{2^{r+1}}.$$

Since, by Lemma 3.2.1, the expression $\mathfrak{T}_2$ modulo $2^{r+1}$ will depend on the value of $\rho \pmod{2^{r+1}}$ we shall once again split this up into two distinct cases.

**Suppose** $\rho \equiv 1 \pmod{2^{r+1}}$:

In this case we simply find,

$$\mathfrak{T}_2 \equiv 2\chi_1 + \chi - \chi - (\chi_1 + \chi_2) + (\chi_2 - \chi_1) \equiv 0 \pmod{2^{r+1}}.$$

This allows us to deduce that, whenever $\rho \equiv 1 \pmod{2^{r+1}}$, we have

$$(-1)^{\frac{(\rho-1)}{2^r}\frac{\mathfrak{T}_2}{2^r}} = 1.$$

**Suppose** $\rho \equiv 2^r + 1 \pmod{2^{r+1}}$:

In order to calculate $\mathfrak{T}_2 \pmod{2^{r+1}}$ in the case that $\rho \equiv 2^r + 1$ we must go through all the possibilities of $\chi_1$, $\chi_2$ and $\chi$ being even or odd. Using Lemma 3.2.1 we find:

$\chi\ \chi_1\ \chi_2$

e e e $\quad \equiv (2^{r-1}+1)(2\chi_1+\chi) - (2^{r-1}+1)\chi$ $\quad \equiv 0$

$\quad -(2^{r-1}+1)(\chi_1+\chi_2) + (2^{r-1}+1)(\chi_2-\chi_1)$

e e o $\quad \equiv (2^{r-1}+1)(2\chi_1+\chi) - (2^{r-1}+1)\chi$ $\quad \equiv 0$

$\quad -(2^r+1)\big((2^{r-1}+1)(\chi_1+\chi_2) - 2^{r-1} - (2^{r-1}+1)(\chi_2-\chi_1) + 2^{r-1}\big)$

e o e $\quad \equiv (2^{r-1}+1)(2\chi_1+\chi) - (2^{r-1}+1)\chi$ $\quad \equiv 0$

$\quad -(2^r+1)\big((2^{r-1}+1)(\chi_1+\chi_2) - 2^{r-1} - (2^{r-1}+1)(\chi_2-\chi_1) + 2^{r-1}\big)$

e o o $\quad \equiv (2^{r-1}+1)(2\chi_1+\chi) - (2^{r-1}+1)\chi$ $\quad \equiv 0$

$\quad -(2^{r-1}+1)(\chi_1+\chi_2) + (2^{r-1}+1)(\chi_2-\chi_1)$

o e e $\quad \equiv (2^r+1)\big((2^{r-1}+1)(2\chi_1+\chi) - 2^{r-1} - (2^{r-1}+1)\chi + 2^{r-1}\big)$ $\quad \equiv 0$

$\quad -(2^{r-1}+1)(\chi_1+\chi_2) + (2^{r-1}+1)(\chi_2-\chi_1)$

o e o $\quad \equiv (2^r+1)\big((2^{r-1}+1)(2\chi_1+\chi) - 2^{r-1} - (2^{r-1}+1)\chi + 2^{r-1}\big)$ $\quad \equiv 0$

$\quad -(2^r+1)\big((2^{r-1}+1)(\chi_1+\chi_2) - 2^{r-1} - (2^{r-1}+1)(\chi_2-\chi_1) + 2^{r-1}\big)$

o o e $\quad \equiv (2^r+1)\big((2^{r-1}+1)(2\chi_1+\chi) - 2^{r-1} - (2^{r-1}+1)\chi + 2^{r-1}\big)$ $\quad \equiv 0$

$\quad -(2^r+1)\big((2^{r-1}+1)(\chi_1+\chi_2) - 2^{r-1} - (2^{r-1}+1)(\chi_2-\chi_1) + 2^{r-1}\big)$

o o o $\quad \equiv (2^r+1)\big((2^{r-1}+1)(2\chi_1+\chi) - 2^{r-1} - (2^{r-1}+1)\chi + 2^{r-1}\big)$ $\quad \equiv 0$

$\quad -(2^{r-1}+1)(\chi_1+\chi_2) + (2^{r-1}+1)(\chi_2-\chi_1)$

Since in every case $\mathfrak{T}_2 \equiv 0 \pmod{2^{r+1}}$ we may once again conclude that, whenever $\rho \equiv 2^r + 1 \pmod{2^{r+1}}$, we have

$$(-1)^{\frac{(\rho-1)}{2^r}\frac{\mathfrak{T}_2}{2^r}} = 1.$$

In conclusion we have shown that the function $\jmath_3$ may be described as,

$$\begin{aligned}
\jmath_3(\alpha_3) &= (-1)^{\frac{1}{2^{2r}}(J_2(\chi_1 : X_3) - (1 - \frac{1}{\rho^{\chi_1}}))} . (-1)^{\frac{1}{2^{2r}}(J_2(\chi_2 : X_3) - (1 - \frac{1}{\rho^{\chi_2}}))} \\
&= (-1)^{\frac{1}{2^{2r}}(J_2(X_1 : X_3) - (1 - \frac{1}{\rho^{X_1}}))} . (-1)^{\frac{1}{2^{2r}}(J_2(X_2 : X_3) - (1 - \frac{1}{\rho^{X_2}}))} \\
&= \jmath_2\left(\begin{pmatrix} \pi^{X_1}a_1 & 0 \\ 0 & \pi^{X_3}a_3 \end{pmatrix}\right) \cdot \jmath_2\left(\begin{pmatrix} \pi^{X_2}a_2 & 0 \\ 0 & \pi^{X_3}a_3 \end{pmatrix}\right) \\
&= \prod_{i=1}^{2} (-1)^{\frac{(\rho-1)}{2^r} X_i \max(X_i, X_3)} (-1)^{\frac{(\rho-1)}{2^r} \frac{\max(X_i, X_3)(\max(X_i, X_3)+1)}{2}},
\end{aligned}$$

which completes our proof. $\square$

### 3.2.3 $\jmath_n$ on the torus $T \subset \mathrm{GL}_n(k_\nu)$

**Theorem 3.2.3** *The function $\jmath_n$ on the torus $T \subset GL_n(k_\nu)$ satisfies,*

$$\begin{aligned}
\jmath_n(\alpha_n) &= \prod_{i=1}^{n-1} \jmath_2\left(\begin{pmatrix} \pi^{X_i}a_i & 0 \\ 0 & \pi^{X_n}a_n \end{pmatrix}\right) \\
&= \prod_{i=1}^{n-1} (-1)^{\frac{(\rho-1)}{2^r} X_i \max(X_i, X_n)} (-1)^{\frac{(\rho-1)}{2^r} \frac{\max(X_i, X_n)(\max(X_i, X_n)+1)}{2}}.
\end{aligned}$$

**PROOF OF THEOREM:**

Looking back to Theorem 3.1.6 we see that we have defined the function $\kappa_n$ in two separate cases $\chi_1 \geq X_n$ and $X_n \geq \chi_1$. Therefore, in order to prove this result we must again consider two cases.

Before we continue let us remind ourselves of the notation defined earlier,

$$\{\chi_1, \chi_2, \cdots, \chi_{n-1}\} = \{X_1, X_2, \ldots, X_{n-1}\} \quad \text{where,} \quad \chi_1 \geq \chi_2 \geq \cdots \geq \chi_{n-1}.$$

• $\chi_1 \geq X_n$ :

If we refer back to Theorem 3.1.6 then we see that when $\chi_1 \geq X_n$ the function $\kappa_n$ is defined inductively. Bearing this in mind and realizing that we have already proved the theorem in the cases $n = 2$ and $n = 3$ we shall now use induction to show that it extends to any dimension $n$.

As we have seen the function $\jmath_n$ has been defined by,

$$\begin{aligned}\jmath_n(\alpha_n) &= (-1)^{\frac{1}{2^{2r}}}\left(J_n(\alpha_n) - (1 - \tfrac{1}{\rho^{X_1+.+X_{n-1}}})\right)\\ &= (-1)^{\frac{1}{2^{2r}}}\left(\kappa_n(I_n) - \kappa_n(\alpha_n) - (1 - \tfrac{1}{\rho^{X_1+.+X_{n-1}}})\right),\end{aligned}$$

where, by Theorem 3.1.6, the function $\kappa_n$ satisfies

$$\kappa_n(\alpha_n) = \frac{1}{\rho^{n\chi_1}} - \frac{(\rho-1)}{\rho^{n\chi_1}(\rho^n-1)} + \frac{1}{\rho^{\chi_1}}\left(\kappa_{n-1}(\chi_2,.,\chi_{n-1}:X_n) - \left(\frac{1}{\rho^{(n-1)\chi_1}} - \frac{(\rho-1)}{\rho^{(n-1)\chi_1}(\rho^{n-1}-1)}\right)\right)$$

$$\begin{aligned}&= \frac{1}{\rho^{n\chi_1}} - \frac{(\rho-1)}{\rho^{n\chi_1}(\rho^n-1)} + \frac{1}{\rho^{\chi_1}}\left(\kappa_{n-1}(I_{n-1}) - \left(\frac{1}{\rho^{(n-1)\chi_1}} - \frac{(\rho-1)}{\rho^{(n-1)\chi_1}(\rho^{n-1}-1)}\right)\right)\\ &\quad - \frac{1}{\rho^{\chi_1}}\left(\kappa_{n-1}(I_{n-1}) - \kappa_{n-1}(\chi_2,.,\chi_{n-1}:X_n) - (1 - \frac{1}{\rho^{\chi_2+.+\chi_{n-1}}})\right) - \frac{1}{\rho^{\chi_1}}\left(1 - \frac{1}{\rho^{\chi_2+.+\chi_{n-1}}}\right)\end{aligned}$$

$$\begin{aligned}&= \frac{1}{\rho^{n\chi_1}} - \frac{(\rho-1)}{\rho^{n\chi_1}(\rho^n-1)} + \frac{1}{\rho^{\chi_1}}\left(\left(1 - \frac{(\rho-1)}{(\rho^{n-1}-1)}\right) - \left(\frac{1}{\rho^{(n-1)\chi_1}} - \frac{(\rho-1)}{\rho^{(n-1)\chi_1}(\rho^{n-1}-1)}\right)\right)\\ &\quad - \frac{1}{\rho^{\chi_1}}\left(\kappa_{n-1}(I_{n-1}) - \kappa_{n-1}(\chi_2,.,\chi_{n-1}:X_n) - (1 - \frac{1}{\rho^{\chi_2+.+\chi_{n-1}}})\right) - \frac{1}{\rho^{\chi_1}}\left(1 - \frac{1}{\rho^{\chi_2+.+\chi_{n-1}}}\right).\end{aligned}$$

Using this result we are able to calculate,

$$J_n(\alpha_n) - (1 - \frac{1}{\rho^{X_1+.+X_{n-1}}}) \quad = \quad \kappa_n(I_n) - \kappa_n(\alpha_n) - (1 - \frac{1}{\rho^{X_1+.+X_{n-1}}})$$

$$= \left(1 - \tfrac{(\rho-1)}{(\rho^n-1)}\right) - \left(\tfrac{1}{\rho^{n\chi_1}} - \tfrac{(\rho-1)}{\rho^{n\chi_1}(\rho^n-1)}\right) - \tfrac{1}{\rho^{\chi_1}}\left(\kappa_{n-1}(I_{n-1}) - \left(\tfrac{1}{\rho^{(n-1)\chi_1}} - \tfrac{(\rho-1)}{\rho^{(n-1)\chi_1}(\rho^{n-1}-1)}\right)\right)$$
$$+ \tfrac{1}{\rho^{\chi_1}}\left(\kappa_{n-1}(I_{n-1}) - \kappa_{n-1}(\chi_2,\cdots,\chi_{n-1} : X_n) - \left(1 - \tfrac{1}{\rho^{\chi_2+.+\chi_{n-1}}}\right)\right)$$
$$+ \tfrac{1}{\rho^{\chi_1}}\left(1 - \tfrac{1}{\rho^{\chi_2+.+\chi_{n-1}}}\right) - \left(1 - \tfrac{1}{\rho^{X_1+.+X_{n-1}}}\right)$$

$$= \left[\tfrac{1}{\rho^{n\chi_1}}(\rho^{n\chi_1} - 1) - \tfrac{(\rho-1)}{\rho^{n\chi_1}(\rho^n-1)}(\rho^{n\chi_1} - 1) - \tfrac{1}{\rho^{n\chi_1}}(\rho^{(n-1)\chi_1} - 1) + \tfrac{(\rho-1)}{\rho^{n\chi_1}(\rho^{n-1}-1)}(\rho^{(n-1)\chi_1} - 1)\right.$$
$$\left. + \tfrac{1}{\rho^{\chi_1+\chi_2+.+\chi_{n-1}}}(\rho^{\chi_2+.+\chi_{n-1}} - 1) - \tfrac{1}{\rho^{X_1+.+X_{n-1}}}(\rho^{X_1+.+X_{n-1}} - 1)\right]$$
$$+ \tfrac{1}{\rho^{\chi_1}}\left[\kappa_{n-1}(I_{n-1}) - \kappa_{n-1}(\chi_2,.,\chi_{n-1} : X_n) - \left(1 - \tfrac{1}{\rho^{\chi_2+.+\chi_{n-1}}}\right)\right]$$

$$= \ (\rho - 1)\left[\tfrac{1}{\rho^{n\chi_1}}\left((\rho^{n\chi_1-1}+.+1) - (\rho^{n(\chi_1-1)}+.+1) - (\rho^{(n-1)\chi_1-1}+.+1) + (\rho^{(n-1)(\chi_1-1)}+.+1)\right)\right.$$
$$\left. - \tfrac{1}{\rho^{X_1+.+X_{n-1}}}(\rho^{(X_1+.+X_{n-1})-1}+.+1) + \tfrac{1}{\rho^{\chi_1+\chi_2+.+\chi_{n-1}}}(\rho^{(\chi_2+.+\chi_{n-1})-1}+.+1)\right]$$
$$+ \tfrac{1}{\rho^{\chi_1}}\left[\kappa_{n-1}(I_{n-1}) - \kappa_{n-1}(\chi_2,\cdots,\chi_{n-1} : X_n) - \left(1 - \tfrac{1}{\rho^{\chi_2+\cdots+\chi_{n-1}}}\right)\right].$$

However, as a consequence of our inductive hypothesis in $(n-1)$-dimensions, modulo 2, we must have

$$\tfrac{1}{2^{2r}}\left[\kappa_{n-1}(I_{n-1}) - \kappa_{n-1}(\alpha_{n-1}) - \left(1 - \tfrac{1}{\rho^{X_1+.+X_{n-2}}}\right)\right] \equiv \tfrac{1}{2^{2r}}\left[\sum_{i=1}^{n-2}\kappa_2(I_2) - \kappa_2(X_i : X_{n-1}) - \left(1 - \tfrac{1}{\rho^{X_i}}\right)\right].$$

Therefore, by defining $\mathfrak{T}_3$ to be the expression

$$\mathfrak{T}_3 := \left[\tfrac{1}{\rho^{n\chi_1}}\ \left((\rho^{n\chi_1-1}+.+1) - (\rho^{n(\chi_1-1)}+.+1) - (\rho^{(n-1)\chi_1-1}+.+1) + (\rho^{(n-1)(\chi_1-1)}+.+1)\right)\right.$$
$$\left. - \tfrac{1}{\rho^{X_1+.+X_{n-1}}}\ (\rho^{(X_1+.+X_{n-1})-1}+.+1) + \tfrac{1}{\rho^{\chi_1+\chi_2+.+\chi_{n-1}}}\ (\rho^{(\chi_2+.+\chi_{n-1})-1}+.+1)\right]$$

and using our inductive hypothesis we may write,

$$\jmath_n(\alpha_n) = (-1)^{\frac{(\rho-1)}{2^r}\frac{\mathfrak{T}_3}{m}}.(-1)^{\frac{1}{2^{2r}}\left(\kappa_{n-1}(I_{n-1})-\kappa_{n-1}(\chi_2,\dots,\chi_{n-1}:X_n)-\left(1-\frac{1}{\rho^{\chi_2+\dots+\chi_{n-1}}}\right)\right)}$$

$$= (-1)^{\frac{(\rho-1)}{2^r}\frac{\mathfrak{T}_3}{2^r}}.(-1)^{\frac{1}{2^{2r}}\left(\sum_{i=2}^{n-1}\kappa_2(I_2)-\kappa_2(\chi_i:X_n)-\left(1-1/\rho^{\chi_i}\right)\right)}$$

$$= (-1)^{\frac{(\rho-1)}{2^r}\frac{\mathfrak{T}_3}{2^r}}.\prod_{i=2}^{n-1}\jmath_2(\chi_i : X_n).$$

So it simply remains for us to calculate the value of $\mathfrak{T}_3 \pmod{2^{r+1}}$. In order to make this easier to follow we shall define $\mathfrak{Z} = \chi_2 + \ldots + \chi_{n-1}$. We are then concerned with finding,

$$\mathfrak{T}_3 = \frac{1}{\rho^{n\chi_1}}\Big[ (\rho^{n\chi_1-1}+.+1) - (\rho^{n(\chi_1-1)}+.+1) - (\rho^{(n-1)\chi_1-1}+.+1) + (\rho^{(n-1)(\chi_1-1)}+.+1)\Big]$$
$$- \frac{1}{\rho^{\chi_1+\mathfrak{Z}}}\Big[ (\rho^{(\chi_1+\mathfrak{Z})-1}+\ldots+1) - (\rho^{(\mathfrak{Z})-1}+\ldots+1)\Big] \pmod{2^{r+1}}.$$

**Suppose** $\rho \equiv 1 \pmod{2^{r+1}}$**:**

Once again using Lemma 3.2.1 we find,

$$\mathfrak{T}_3 \equiv n\chi_1 - \chi_1 - (n-1)\chi_1 + \chi_1 - (\chi_1 + \mathfrak{Z}) + \mathfrak{Z} \equiv 0 \pmod{2^{r+1}}.$$

Therefore we may conclude that, whenever $\rho \equiv 1 \pmod{2^{r+1}}$, we have

$$(-1)^{\frac{(\rho-1)}{2^r}\frac{\mathfrak{T}_3}{2^r}} = 1.$$

**Suppose** $\rho \equiv 2^r + 1 \pmod{2^{r+1}}$**:**

In order to tackle this we shall first split it into the two distinct cases when $n$ is even and odd. We shall then consider the value of $\mathfrak{T}_3$ modulo $2^{r+1}$ in each of the possible cases when $\chi_1$ and $\mathfrak{Z}$ are even or odd.

**Suppose $n$ is even**

Using Lemma 3.2.1 we find that, modulo $2^{r+1}$, $\mathfrak{T}_3$ is congruent to:

$$\begin{array}{lll}
\chi_1 & 3 & \\
\text{e} & \text{e:} & \equiv (2^{r-1}+1)n\chi_1 - \chi_1 - (2^{r-1}+1)(n-1)\chi_1 + (2^{r-1}+1)\chi_1 \\
& & \quad -(2^{r-1}+1)(\chi_1+3) + (2^{r-1}+1)3 \\
& & \equiv 2^{r-1}\chi_1 \\
& & \\
\text{e} & \text{o:} & \equiv (2^{r-1}+1)n\chi_1 - \chi_1 - (2^{r-1}+1)(n-1)\chi_1 + (2^{r-1}+1)\chi_1 \\
& & \quad -(2^r+1)((2^{r-1}+1)(\chi_1+3) - 2^{r-1}) + (2^r+1)((2^{r-1}+1)3 - 2^{r-1}) \\
& & \equiv -2^{r-1}\chi_1(1+2^r) \\
& & \\
\text{o} & \text{e:} & \equiv (2^{r-1}+1)n\chi_1 - \chi_1 - (2^{r-1}+1)(n-1)\chi_1 + 2^{r-1} + (2^{r-1}+1)\chi_1 \\
& & \quad -2^{r-1} - (2^r+1)((2^{r-1}+1)(\chi_1+3) - 2^{r-1}) + (2^r+1)(2^{r-1}+1)3 \\
& & \equiv -2^{r-1}(1+2^r)(\chi_1-1) \\
& & \\
\text{o} & \text{o:} & \equiv (2^{r-1}+1)n\chi_1 - \chi_1 - (2^{r-1}+1)(n-1)\chi_1 + 2^{r-1} + (2^{r-1}+1)\chi_1 \\
& & \quad -2^{r-1} - (2^{r-1}+1)(\chi_1+3) + (2^{r-1}+1)3 - 2^{r-1} \\
& & \equiv 2^{r-1}(\chi_1-1)
\end{array}$$

We complete this section by dividing out the factor $2^{r-1}$ from the original expression $\mathfrak{T}_3/2^r$ and considering the value of what remains modulo 4.

$$\begin{array}{llll}
\chi_1 & 3 & & \\
\text{e} & \text{e:} & \equiv \chi_1 & \\
\text{e} & \text{o:} & \equiv -\chi_1(1+2^r) & \equiv \begin{cases} -\chi_1 & r>1 \\ \chi_1 & r=1 \end{cases} \\
\text{o} & \text{e:} & \equiv -(1+2^r)(\chi_1-1) & \equiv \begin{cases} -(\chi_1-1) & r>1 \\ (\chi_1-1) & r=1 \end{cases} \\
\text{o} & \text{o:} & \equiv (\chi_1-1) &
\end{array}$$

Since we are not interested in the overall sign of the expression, referring back to the proof of Theorem 3.2.1, we are able to summarise our results using the single formula,

$$\begin{aligned}
(-1)^{\frac{(\rho-1)}{m}\frac{\mathfrak{T}_3}{2^r}} &= (-1)^{\frac{(\rho-1)}{2^r}\frac{\chi_1(\chi_1-1)}{2}} \\
&= \jmath_2(\chi_1 : X_n), \quad \text{since } \chi_1 \geq X_n.
\end{aligned}$$

**Suppose $n$ is odd**

Once again using Lemma 3.2.1 we find that, modulo $2^{r+1}$, $\mathfrak{T}_3$ is congruent to:

$$\begin{array}{lll}
\chi_1 & 3 & \\
\mathrm{e} & \mathrm{e:} & \equiv (2^{r-1}+1)n\chi_1 - (2^{r-1}+1)\chi_1 - (2^{r-1}+1)(n-1)\chi_1 + \chi_1 \\
 & & \quad -(2^{r-1}+1)(\chi_1+3) + (2^{r-1}+1)3 \\
 & & \equiv -2^{r-1}\chi_1 \\
 & & \\
\mathrm{e} & \mathrm{o:} & \equiv (2^{r-1}+1)n\chi_1 - (2^{r-1}+1)\chi_1 - (2^{r-1}+1)(n-1)\chi_1 + \chi_1 \\
 & & \quad -(2^r+1)((2^{r-1}+1)(\chi_1+3) - 2^{r-1}) + (2^r+1)((2^{r-1}+1)3 - 2^{r-1}) \\
 & & \equiv -2^{r-1}\chi_1(3+2^r) \\
 & & \\
\mathrm{o} & \mathrm{e:} & \equiv (2^r+1)((2^{r-1}+1)n\chi_1 - 2^{r-1} - (2^{r-1}+1)\chi_1 + 2^{r-1} - (2^{r-1}+1)(n-1)\chi_1 + \chi_1) \\
 & & \quad -(2^r+1)((2^{r-1}+1)(\chi_1+3) - 2^{r-1}) + (2^r+1)(2^{r-1}+1)3 \\
 & & \equiv -2^{r-1}((\chi_1-1) + 2^r(\chi_1-1)) \\
 & & \\
\mathrm{o} & \mathrm{o:} & \equiv (2^r+1)((2^{r-1}+1)n\chi_1 - 2^{r-1} - (2^{r-1}+1)\chi_1 + 2^{r-1} - (2^{r-1}+1)(n-1)\chi_1 + \chi_1) \\
 & & \quad -(2^{r-1}+1)(\chi_1+3) + (2^{r-1}+1)3 - 2^{r-1} \\
 & & \equiv 2^{r-1}(3\chi_1+1).
\end{array}$$

Dividing out the factor $2^{r-1}$ from the expression $\mathfrak{T}_3/2^r$ and considering the value of what remains modulo 4 we find,

$$\begin{array}{lllll}
\chi_1 & 3 & & & \\
\mathrm{e} & \mathrm{e:} & \equiv -\chi_1 & & \\
\mathrm{e} & \mathrm{o:} & \equiv -\chi_1(3+2^r) & \equiv & \begin{cases} \chi_1 & r>1 \\ -\chi_1 & r=1 \end{cases} \\
\mathrm{o} & \mathrm{e:} & \equiv -((\chi_1-1)+2^r(\chi_1-1)) & \equiv & \begin{cases} -(\chi_1-1) & r>1 \\ (\chi_1-1) & r=1 \end{cases} \\
\mathrm{o} & \mathrm{o:} & \equiv -(-\chi_1+1) & &
\end{array}$$

Once again, by ignoring the overall sign of the expression, we may express these results by the single equation,

$$\begin{aligned}
(-1)^{\frac{(\rho-1)}{m}\frac{\mathfrak{T}_3}{2^r}} &= (-1)^{\frac{(\rho-1)}{2^r}\frac{\chi_1(\chi_1-1)}{2}} \\
&= \jmath_2(\chi_1 : X_n) \qquad \text{since} \quad \chi_1 \geq X_n.
\end{aligned}$$

Finally we are able to conclude that the function $\jmath_n$ does indeed satisfy,

$$\begin{aligned}\jmath_n(\alpha_n) &= \jmath_2(\chi_1 : X_n).\prod_{i=2}^{n-1}\jmath_2(\chi_i : X_n) = \prod_{i=1}^{n-1}\jmath_2(\chi_i : X_n)\\ &= \prod_{i=1}^{n-1}\jmath_2(X_i : X_n) \qquad \text{since } \{\chi_1, ., \chi_{n-1}\} = \{X_1, ., X_{n-1}\}\\ &= \prod_{i=1}^{n-1}\jmath_2\left(\begin{pmatrix}\pi^{X_i}a_i & 0\\ 0 & \pi^{X_n}a_n\end{pmatrix}\right).\end{aligned}$$

Therefore, whenever $\chi_1 \geq X_n$, the original statement of our theorem is indeed true for any dimension $n$.

• $X_n \geq \chi_1$ :

Referring back to page 56, whenever $X_n \geq \chi_1$, we have been able to show that the function $\kappa_n$ satisfies,

$$\kappa_n(\alpha_n) = \frac{1}{\rho^{(X_1+\ldots+X_n)}} - \frac{(\rho-1)}{\rho^{nX_n}(\rho^n-1)}.$$

Therefore we are able to calculate,

$$J_n(\alpha_n) - (1 - \frac{1}{\rho^{X_1+.+X_{n-1}}}) = \kappa_n(I_n) - \kappa_n(\alpha_n) - \left(1 - \frac{1}{\rho^{X_1+.+X_{n-1}}}\right)$$

$$= \left(1 - \frac{(\rho-1)}{(\rho^n-1)}\right) - \left(\frac{1}{\rho^{(X_1+\ldots+X_n)}} - \frac{(\rho-1)}{\rho^{nX_n}(\rho^n-1)}\right) - \frac{1}{\rho^{X_1+.+X_{n-1}}}(\rho^{X_1+.+X_{n-1}}-1)$$

$$= -\frac{(\rho-1)}{\rho^{nX_n}(\rho^n-1)}(\rho^{nX_n}-1) + \frac{1}{\rho^{X_1+.+X_n}}(\rho^{X_1+.+X_n}-1) - \frac{1}{\rho^{X_1+.+X_{n-1}}}(\rho^{X_1+.+X_{n-1}}-1)$$

$$= (\rho-1)\left[-\frac{(\rho^{n(X_n-1)}+.+1)}{\rho^{nX_n}} + \frac{(\rho^{(X_1+.+X_n)-1}+.+1)}{\rho^{X_1+.+X_n}} - \frac{(\rho^{(X_1+.+X_{n-1})-1}+.+1)}{\rho^{X_1+.+X_{n-1}}}\right].$$

Let us now re-define $\mathfrak{Z} := X_1+\ldots+X_{n-1}$. Then we see that the proof of this theorem reduces to the problem of calculating the expression $\mathfrak{T}_4 \pmod{2^{r+1}}$ where,

$$\mathfrak{T}_4 = \left[-\frac{1}{\rho^{nX_n}}(\rho^{n(X_n-1)}+\ldots+1) + \frac{1}{\rho^{\mathfrak{Z}+X_n}}(\rho^{(\mathfrak{Z}+X_n)-1}+\ldots+1) - \frac{1}{\rho^{\mathfrak{Z}}}(\rho^{\mathfrak{Z}-1}+\ldots+1)\right].$$

**Suppose** $\rho \equiv 1 \pmod{2^{r+1}}$**:**

Using Lemma 3.2.1 we find,

$$\mathfrak{T}_4 \equiv -X_n + (3 + X_n) - 3 \equiv 0 \pmod{2^{r+1}}.$$

Therefore whenever $\rho \equiv 1 \pmod{2^{r+1}}$ we have,

$$\jmath_n(\alpha_n) = (-1)^{\frac{(\rho-1)}{2^r}\frac{\mathfrak{T}_4}{2^r}} = 1.$$

**Suppose** $\rho \equiv 2^r + 1 \pmod{2^{r+1}}$**:**

In this case the value of the expression $\mathfrak{T}_4$ modulo $2^{r+1}$ will again depend on whether $n$ is even or odd. Therefore we shall split this into two cases and then consider each of the remaining possibilities.

**Suppose** $n$ **is even**

By Lemma 3.2.1 we calculate the expression $\mathfrak{T}_4$ modulo $2^{r+1}$ to be:

$$
\begin{array}{lll}
X_n & 3 & \\
\mathrm{e} & \mathrm{e} & \equiv -X_n + (2^{r-1}+1)(X_n+3) - (2^{r-1}+1)3 \\
 & & \equiv 2^{r-1}X_n \\
 & & \\
\mathrm{e} & \mathrm{o} & \equiv -X_n + (2^r+1)((2^{r-1}+1)(3+X_n) - 2^{r-1}) - (2^r+1)((2^{r-1}+1)3 - 2^{r-1}) \\
 & & \equiv 2^{r-1}X_n(3+2^r) \\
 & & \\
\mathrm{o} & \mathrm{e} & \equiv -X_n + (2^r+1)((2^{r-1}+1)(3+X_n) - 2^{r-1}) - (2^{r-1}+1)3 \\
 & & \equiv 2^{r-1}X_n(3+2^r) + 2^{r-1}3(2+2^r) - 2^{r-1}(1+2^r) \\
 & & \\
\mathrm{o} & \mathrm{o} & \equiv -X_n + (2^{r-1}+1)(X_n+3) - (2^r+1)((2^{r-1}+1)3 - 2^{r-1}) \\
 & & \equiv 2^{r-1}X_n - 2^{r-1}3(2+2^r) + 2^{r-1}(1+2^r).
\end{array}
$$

Once again, since we are interested in $\mathfrak{T}_4/2^r \pmod 2$ we may divide out the factor $2^{r-1}$ and then concentrate on calculating what remains modulo 4. Disregarding the sign of the overall expression we find,

$$
\begin{array}{lll}
X_n & \mathcal{Z} & \\
\mathrm{e} & \mathrm{e} & \equiv X_n \\
 & & \\
\mathrm{e} & \mathrm{o} & \equiv X_n(3+2^r) \equiv X_n \\
\mathrm{o} & \mathrm{e} & \equiv X_n(3+2^r)+\mathcal{Z}(2+2^r)-(1+2^r) \equiv \begin{cases} (X_n+2\mathcal{Z}+1) & r>1 \\ X_n+1 \equiv (X_n+2\mathcal{Z}+1) & r=1, \end{cases} \\
 & & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{as } \mathcal{Z} \text{ is even.} \\
\mathrm{o} & \mathrm{o} & \equiv X_n-\mathcal{Z}(2+2^r)+(1+2^r) \qquad\quad \equiv \begin{cases} (X_n+2\mathcal{Z}+1) & r>1 \\ X_n-1 \equiv (X_n+2\mathcal{Z}+1) & r=1, \end{cases} \\
 & & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{as } \mathcal{Z} \text{ is odd.}
\end{array}
$$

So, when $n$ is even, the function $\jmath_n$ may be described by,

$$
\begin{aligned}
\jmath_n(\alpha_n) &= (-1)^{\frac{(p-1)}{2^r}\mathcal{Z}X_n}(-1)^{\frac{(p-1)}{2^r}\frac{X_n(X_n+1)}{2}} \\
&= (-1)^{\frac{(p-1)}{2^r}(X_1+\ldots+X_{n-1})X_n}(-1)^{\frac{(p-1)}{2^r}\frac{X_n(X_n+1)}{2}}.
\end{aligned}
$$

### Suppose $n$ is odd

In this case, when going through the possibilities, the expression $\mathfrak{T}_4$ modulo $2^{r+1}$ is found to be:

$$
\begin{array}{lll}
X_n & \mathcal{Z} & \\
\mathrm{e} & \mathrm{e} & \equiv -(2^{r-1}+1)X_n+(2^{r-1}+1)(X_n+\mathcal{Z})-(2^{r-1}+1)\mathcal{Z} \\
 & & \equiv 0 \\
 & & \\
\mathrm{e} & \mathrm{o} & \equiv -(2^{r-1}+1)X_n+(2^r+1)((2^{r-1}+1)(\mathcal{Z}+X_n)-2^{r-1})-(2^r+1)((2^{r-1}+1)\mathcal{Z}-2^{r-1}) \\
 & & \equiv 2^{r-1}(2+2^r)X_n \\
 & & \\
\mathrm{o} & \mathrm{e} & \equiv -(2^r+1)((2^{r-1}+1)X_n-2^{r-1})+(2^r+1)((2^{r-1}+1)(\mathcal{Z}+X_n)-2^{r-1})-(2^{r-1}+1)\mathcal{Z} \\
 & & \equiv 2^{r-1}(2+2^r)\mathcal{Z} \\
 & & \\
\mathrm{o} & \mathrm{o} & \equiv -(2^r+1)((2^{r-1}+1)X_n-2^{r-1})+(2^{r-1}+1)(X_n+\mathcal{Z})-(2^r+1)((2^{r-1}+1)\mathcal{Z}-2^{r-1}) \\
 & & \equiv -2^{r-1}(2+2^r)X_n-2^{r-1}(2+2^r)\mathcal{Z}+2^{r-1}(2+2^{r+1}).
\end{array}
$$

When we divide out the factor $2^{r-1}$ and look at what remains modulo 4 in this case we simply find,

$$
\begin{array}{lll l}
X_n & \mathfrak{Z} & & \\
\text{e} & \text{e} & \equiv 0 & \\
\\
\text{e} & \text{o} & \equiv (2+2^r)X_n \equiv 0 & \text{since } X_n \text{ is even} \\
\\
\text{o} & \text{e} & \equiv (2+2^r)\mathfrak{Z} \equiv 0 & \text{since } \mathfrak{Z} \text{ is even} \\
\\
\text{o} & \text{o} & \equiv -(2+2^r)X_n - (2+2^r)\mathfrak{Z} + (2+2^{r+1}) \equiv 2 & \text{since } X_n + \mathfrak{Z} \text{ is even.}
\end{array}
$$

After a moment's thought we see that these results may be expressed as,

$$
\begin{aligned}
\jmath_n(\alpha_n) &= (-1)^{\frac{(\rho-1)}{2^r}\mathfrak{Z}X_n} \\
&= (-1)^{\frac{(\rho-1)}{2^r}(X_1+.+X_{n-1})X_n},
\end{aligned}
$$

since this does indeed give us the required solutions.

Finally, we have shown that whenever $X_n \geq \chi_1$ the function $\jmath_n$ satisfies,

$$
\begin{aligned}
\jmath_n(\alpha_n) &= \begin{cases} (-1)^{\frac{(\rho-1)}{2^r}(X_1+.+X_{n-1})X_n}(-1)^{\frac{(\rho-1)}{2^r}\frac{X_n(X_n+1)}{2}} & \text{whenever } n \text{ is even} \\ \\ (-1)^{\frac{(\rho-1)}{2^r}(X_1+.+X_{n-1})X_n} & \text{whenever } n \text{ is odd.} \end{cases} \\
&= (-1)^{\frac{(\rho-1)}{2^r}(X_1+.+X_{n-1})X_n}\left((-1)^{\frac{(\rho-1)}{2^r}\frac{X_n(X_n+1)}{2}}\right)^{n-1} \\
&= \prod_{i=1}^{n-1}(-1)^{\frac{(\rho-1)}{2^r}X_iX_n}(-1)^{\frac{(\rho-1)}{2^r}\frac{X_n(X_n+1)}{2}} \\
&= \prod_{i=1}^{n-1}(-1)^{\frac{(\rho-1)}{2^r}X_1\max(X_i,X_n)}(-1)^{\frac{(\rho-1)}{2^r}\frac{\max(X_i,X_n)(\max(X_i,X_n)+1)}{2}} \\
&= \prod_{i=1}^{n-1}\jmath_2\left(\begin{pmatrix} \pi^{X_i}a_i & 0 \\ 0 & \pi^{X_n}a_n \end{pmatrix}\right),
\end{aligned}
$$

at which point our proof is complete.

$\square$

## 3.3 The coboundaries $\partial\tau_n$

We are now able to state the main theorem in this chapter.

**Theorem 3.3.1** *Let $k_\nu$, $\pi$ be as previously defined. Then for each $\alpha_n \in T \subset GL_n(k_\nu)$ the cochain $\tau_n$ satisfies,*

$$\begin{aligned}
\tau_n(\alpha_n) &= \jmath_2(\alpha_2)\jmath_3(\alpha_3)\dots\jmath_n(\alpha_n) \\
&= \prod_{\ell=2}^{n}\left( \prod_{k=1}^{\ell-1} \jmath_2\left(\begin{pmatrix} \pi^{X_k}a_1 & 0 \\ 0 & \pi^{X_\ell}a_\ell \end{pmatrix}\right) \right) \\
&= \prod_{(i,j)\in\Phi^+} \jmath_2\left(\begin{pmatrix} \pi^{X_i}a_i & 0 \\ 0 & \pi^{X_j}a_j \end{pmatrix}\right) \\
&= \prod_{(i,j)\in\Phi^+} (-1)^{\frac{(\rho-1)}{2^r}X_i \max(X_i,X_j)}(-1)^{\frac{(\rho-1)}{2^r}\frac{\max(X_i,X_j)(\max(X_i,X_j)+1)}{2}}.
\end{aligned}$$

*where $(-1)$ and $\tau_n$ are both trivial if $m$ is odd.*

**PROOF OF THEOREM:**

The proof of this theorem follows immediately from our earlier results concerning the function $\jmath_n$.

□

# Chapter 4

# Calculation of $dec_\nu$ on the Monomials

## 4.1 Introduction

In this chapter we shall study the cocycle $dec_\nu$ on the group of monomials $M$ in $\mathrm{GL}_n(k_\nu)$. We shall begin by considering $M = T.W$ where, as in the introduction, $T$ is the torus and $W \subset M$ is defined to be the group of permutation matrices. Once again we let $k_\nu$ be a local field with valuation $\nu$ and fixed uniformizing element $\pi \in \mathfrak{O}_\nu$.

### Root Spaces

Previously we have defined the set $\Phi$ to be the roots of $\mathrm{GL}_n(k_\nu)$ relative to $T$,

$$\Phi = \{(i,j) : 1 \leq i,\, j \leq n, i \neq j\}.$$

The set of positive roots $\Phi^+$ has an ordered base of simple roots,

$$\Delta = \{\varsigma = (i, i+1) \in \Phi : 1 \leq i < n\}.$$

We have also seen that the action of $M$ on $T$ induces an action on $\Phi$ which leads to a well defined faithful action of $W$ on $\Phi$ which, for each $w \in W$, satisfies

$$w\varsigma = w(i,j) = (w(i), w(j)).$$

Therefore, each $w \in W$ may be regarded as a permutation on $\Phi$ with the weight space of $w$ defined to be the set,

$$\begin{aligned}\Phi(w) &= \{\varsigma \in \Phi^+ : \, w\varsigma \notin \Phi^+\} \\ &= \{(i,j) \in \Phi : \, i < j, \, w(i) > w(j)\}.\end{aligned}$$

Since $W \cong S_n = \langle (i,i+1) : 1 \leq i < n \rangle$, it may be generated by the simple reflections

$$\{s_\varsigma = s_{(i,i+1)} : \varsigma = (i,i+1) \in \Delta\}.$$

Therefore, each $w \in W$ may be expressed as a minimal product,

$$w = s_{\varsigma_1} \dots s_{\varsigma_\ell}, \qquad \text{with } \varsigma_k = (i_k, i_k+1) \in \Delta,$$

called a *reduced expression* for $w$ where $l(w) = \ell$ is defined to be the length of $w$.

We shall now have a closer look at the action of $W$ on $\Phi$. In particular we shall need some well known results concerning root systems.

**Lemma 2** *For each generator $s_\varsigma = s_{(i,i+1)} \in W$ we have,*

$$\Phi(s_\varsigma) = \{\varsigma\} = \{(i,i+1)\}.$$

**Lemma 3** *For each $w \in W$ and each generator $s_\varsigma = s_{(i,i+1)} \in W$ we have,*

$$|\Phi(ws_\varsigma)| = \begin{cases} |\Phi(w)| + 1 & w(\varsigma) \in \Phi^+, \\ |\Phi(w)| - 1 & w(\varsigma) \in \Phi^-. \end{cases}$$

**Lemma 4** *Let $w = s_{\varsigma_1} \dots s_{\varsigma_\ell}$ be a reduced expression for $w \in W$. Then we have,*

$$|\Phi(w)| = l(w) = \ell.$$

The proof of these three lemmas along with more results concerning root spaces and Coxeter Groups can be found in Carter [2]. Finally, putting these results together we are able to deduce the following:

**Lemma 4.1.1** *Let $w = s_{\varsigma_1} \dots s_{\varsigma_\ell}$ be some reduced expression for $w \in W$ where $\varsigma_k = (i_k, i_k+1) \in \Delta$. Then we have,*

$$\begin{aligned} \Phi(w) &= \{(i,j) \in \Phi^+ : w(i) > w(j)\} \\ &= \{\varsigma_\ell, s_{\varsigma_\ell}(\varsigma_{\ell-1}), s_{\varsigma_\ell} s_{\varsigma_{\ell-1}}(\varsigma_{\ell-2}), \dots, s_{\varsigma_\ell} \dots s_{\varsigma_2}(\varsigma_1)\} \\ &= \{(i_\ell, i_\ell+1), s_{\varsigma_\ell}(i_{\ell-1}, i_{\ell-1}+1), \dots, s_{\varsigma_\ell} \dots s_{\varsigma_2}(i_1, i_1+1)\}. \end{aligned}$$

*Hence, for all $w_1, w_2 \in W$ such that $l(w_1w_2) = l(w_1) + l(w_2)$, we have*

$$\Phi(w_1w_2) = \Phi(w_1)^{w_2^{-1}} \cup \Phi(w_2), \quad \textit{where the union is disjoint.}$$

**PROOF:** This follows directly from the previous three lemmas. □

## 4.2 The cocycle $dec_\nu$ on the monomials $M = T.W$

**Theorem 4.2.1** *Let $k_\nu$ be a local field with valuation $\nu$. Then for each $g \in GL_n(k_\nu)$ and each $w \in W$ the cocycle $dec_\nu$ satisfies,*

$$dec_\nu(g, w) = 1. \tag{4.1}$$

**PROOF OF THEOREM:**

Since we know that $W \subset \mathrm{GL}_n(\mathfrak{O}_\nu)$ the proof of this statement follows directly from Theorem 1.4.1 (page 26) as given in the introduction. □

Furthermore, by applying the cocycle rule, we are able to deduce the following:

**Corollary 1** *For each $g_1$, $g_2 \in GL_n(k_\nu)$ and each $w \in W$ the cocycle $dec_\nu$ satisfies,*

$$dec_\nu(g_1, g_2 w) = dec_\nu(g_1, g_2).$$

## 4.3 The cocycle $dec_\nu$ on $M \subset \mathrm{GL}_2(k_\nu)$

We shall now consider the cocycle $dec_\nu$ on the full group of monomials $M$ in $\mathrm{GL}_2(k_\nu)$. Since we are only dealing with $\mathrm{GL}_2$ we may simply write,

$$W = \{I, \psi\} = \{I, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\} \cong S_2 \qquad \text{where, } \psi = s_{(1,2)}.$$

**Theorem 4.3.1** *Let $k_\nu$ be a local field with valuation $\nu$. Then for any diagonal matrix $\alpha_2 \in T$ the cocycle $dec_\nu$ satisfies,*

$$dec_\nu(\psi, \alpha_2) = dec_\nu(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \alpha_2) = (-1)^{\frac{(\rho-1)}{2^r} \max\{X_1, X_2\}},$$

*where $(-1) = 1$ whenever $m = t$ is odd.*

*Hence, on the full group of monomials in $GL_2(k_\nu)$ the cocycle $dec_\nu$ satisfies,*

$$dec_\nu(\alpha_2\psi, \beta_2\psi) = (-1)^{\frac{(\rho-1)}{2^r} \max\{X_1, X_2\}}\, dec_\nu(\alpha_2, {}^{\psi}\beta_2),$$

*for all $\alpha_2$, $\beta_2 \in T \subset GL_2(k_\nu)$.*

**PROOF OF THEOREM:**

Let us consider,

$$dec_\nu(\psi,\alpha_2) = \prod_{\xi\in\mu_m} \xi^{\int_{k_\nu{}^2\backslash 0} f\left(\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\mathcal{X}\right) f(\xi\mathcal{X})\left(M(\mathcal{X})-M(\alpha_2^{-1}\mathcal{X})\right)\mathrm{d}\mathcal{X}}$$

$$= \prod_{\xi\in\mu_m} \xi^{\int_{k_\nu{}^2\backslash 0} f\binom{x_2}{x_1} f\binom{\xi x_1}{\xi x_2}\left(M(\mathcal{X})-M(\alpha_2^{-1}\mathcal{X})\right)\mathrm{d}\mathcal{X}} =: \prod_{\xi\in\mu_m} \xi^{I(\xi)}.$$

In order to calculate the integral $I(\xi)$ we must first decompose $k_\nu{}^2\backslash 0$ into three disjoint open sets,

$$A_1 = \{\mathcal{X} \in k_\nu{}^2 \setminus 0 : |x_1|_\nu > |x_2|_\nu\}$$
$$A_2 = \{\mathcal{X} \in k_\nu{}^2 \setminus 0 : |x_1|_\nu = |x_2|_\nu\}$$
$$A_3 = \{\mathcal{X} \in k_\nu{}^2 \setminus 0 : |x_1|_\nu < |x_2|_\nu\},$$

and then calculate the integral $I_i(\xi)$, over each of the sets $A_i$, separately. Considering the value of our function $f$ on each of these sets $A_i$ we find,

$$\mathcal{X} \in A_1 \quad \Rightarrow \quad f\binom{x_2}{x_1} f\binom{\xi x_1}{\xi x_2} = f(x_1)f(\xi x_1)$$
$$\mathcal{X} \in A_2 \quad \Rightarrow \quad f\binom{x_2}{x_1} f\binom{\xi x_1}{\xi x_2} = f(x_2)f(\xi x_1)$$
$$\mathcal{X} \in A_3 \quad \Rightarrow \quad f\binom{x_2}{x_1} f\binom{\xi x_1}{\xi x_2} = f(x_2)f(\xi x_2).$$

### The integrals $I_1(\xi)$ and $I_3(\xi)$

Since the function $f : k_\nu\backslash 0 \to \mathbb{Z}$, given in section 1.4.3, is defined to be the characteristic function of a fundamental domain for the action of $\mu_m$ on $k_\nu\backslash 0$ we must have,

$$f(x)f(\xi x) = 0, \qquad \forall x \in k_\nu\backslash 0,\ 1 \neq \xi \in \mu_m.$$

Hence we are able to conclude that, for each $\xi \neq 1$, we have

$$I_1(\xi) = \int_{A_1} f(x_1)f(\xi x_1)\left(M(\mathcal{X}) - M(\alpha_2^{-1}\mathcal{X})\right)\mathrm{d}\mathcal{X} = 0$$
$$I_3(\xi) = \int_{A_3} f(x_2)f(\xi x_2)\left(M(\mathcal{X}) - M(\alpha_2^{-1}\mathcal{X})\right)\mathrm{d}\mathcal{X} = 0.$$

### The integral $I_2(\xi)$

Finally we must calculate the integral,

$$I_2(\xi) = \int_{A_2} f(x_2) f(\xi x_1)\big(M(\mathcal{X}) - M(\alpha_2^{-1}\mathcal{X})\big)\mathrm{d}\mathcal{X}.$$

If we refer back to page 33, in Chapter 2, we see that we have an action of $\mu_m \oplus \mu_m$ on $k_\nu \oplus k_\nu$ such that,

$$A_2 \quad \text{and} \quad \big(M(\mathcal{X}) - M(\alpha_2^{-1}\mathcal{X})\big)\mathrm{d}\mathcal{X}$$

are both $\mu_m \oplus \mu_m$-invariant.

This enables us to write,

$$\begin{aligned} m^2 I_2(\xi) &= \sum_{\xi_1,\xi_2 \in \mu_m} \int_{A_2} f(\xi_1 x_2) f(\xi_2 \xi x_1)\big(M(\mathcal{X}) - M(\alpha_2^{-1}\mathcal{X})\big)\mathrm{d}\mathcal{X} \\ &= \int_{A_2} \big(M(\mathcal{X}) - M(\alpha_2^{-1}\mathcal{X})\big)\mathrm{d}\mathcal{X} =: m^2 I_2. \end{aligned}$$

Since this expression has no dependence on $\xi$ we let $I_2 := I_2(\xi)$. Then we may deduce that,

$$dec_\nu(\psi, \alpha_2) = \prod_{\xi \in \mu_m} \xi^{I_2(\xi)} = \Big(\prod_{\xi \in \mu_m} \xi\Big)^{I_2} = (-1)^{I_2},$$

is trivial whenever $m$ is odd. If however, $m = 2^r$ is even, it still remains to calculate the integral $I_2$.

We shall begin by considering matrices $\alpha_2 \in T$ such that the exponents $X_1$ and $X_2$ are positive. If we note that the space we are integrating over is symmetric in the variables $x_1$ and $x_2$, by defining

$$\chi_1 = \max\{X_1, X_2\} \quad \text{and} \quad \chi_2 = \min\{\mathrm{X}_1, \mathrm{X}_2\},$$

we are left with just one possible case.

That is, $\chi_1 \geq \chi_2 \geq 0$:

$$\begin{aligned} I_2 &= \frac{1}{2^{2r}} \sum_{n=0}^{\chi_1 - 1} \left(\frac{1}{\rho^n} - \frac{1}{\rho^{n+1}}\right)^2 = \frac{(\rho-1)^2}{2^{2r}} \frac{1}{\rho^2} \sum_{n=0}^{\chi_1-1} \frac{1}{\rho^{2n}} \\ &= \left(\frac{(\rho-1)}{2^r}\right)^2 \frac{1}{\rho^{2\chi_1}} (\rho^{2(\chi_1 - 1)} + \ldots + 1) \equiv \frac{(\rho-1)}{2^r}\chi_1 \pmod 2. \end{aligned}$$

So, whenever $X_1,\ X_2 \geq 0$, we have found that the cocycle satisfies,

$$dec_\nu(\psi, \alpha_2) = dec_\nu\left(\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \alpha_2\right) = (-1)^{\frac{(\rho-1)}{2^r} \max\{X_1, X_2\}}.$$

Let us now consider the matrices $\alpha_2 \in T$ such that at least one of the exponents $X_1$ or $X_2$ is negative. Then there exists a positive integer $N \in \mathbb{Z}$ large enough such that $N + X_i \geq 0$ for each $i$.

Using the cocycle rule and the previous equation we now find,

$$\begin{aligned} dec_\nu(\psi, \pi^N \alpha_2) &= \frac{dec_\nu(\psi\pi^N, \alpha_2)\, dec_\nu(\psi, \pi^N)}{dec_\nu(\pi^N, \alpha_2)} \\ &= dec_\nu(\psi, \alpha_2)\, dec_\nu(\psi, \pi^N) \\ &= dec_\nu(\psi, \alpha_2)(-1)^{\frac{(\rho-1)}{2^r}N}. \end{aligned}$$

However this allows us to calculate,

$$\begin{aligned} dec_\nu(\psi, \alpha_2) &= dec_\nu(\psi, \pi^N \alpha_2)(-1)^{\frac{(\rho-1)}{2^r}N} \\ &= (-1)^{\frac{(\rho-1)}{2^r}\max\{N+X_1, N+X_2\}}(-1)^{\frac{(\rho-1)}{2^r}N} \\ &= (-1)^{\frac{(\rho-1)}{2^r}(N+\max\{X_1,X_2\})}(-1)^{\frac{(\rho-1)}{2^r}N} \\ &= (-1)^{\frac{(\rho-1)}{2^r}\max\{X_1,X_2\}}. \end{aligned}$$

Therefore our result is in fact true for any diagonal matrix $\alpha_2 \in T \subset \mathrm{GL}_2(k_\nu)$.

Finally, using the cocycle rule, we may now conclude that on the monomials $M$ in $\mathrm{GL}_2(k_\nu)$ the 2-cocycle $dec_\nu$ satisfies,

$$\begin{aligned} dec_\nu(\alpha_2\psi, \beta_2\psi) &= dec_\nu(\alpha_2\psi, \beta_2) \\ &= dec_\nu(\alpha_2, \psi\beta_2)\, dec_\nu(\psi, \beta_2) \\ &= dec_\nu(\alpha_2, \psi\beta_2\psi)\, dec_\nu(\psi, \beta_2) \\ &= (-1)^{\frac{(\rho-1)}{2^r}\max\{Y_1,Y_2\}}\, dec_\nu(\alpha_2, {}^{\psi}\beta_2). \end{aligned}$$

$\square$

## 4.4 The cocycle $dec_\nu$ on $M \subset \mathrm{GL}_3(k_\nu)$

Before we begin this section let us note that we now have $W \cong S_3 = D_6$. Let us extend the notation of the previous section by defining,

$$\psi = s_{(1,2)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \varphi = s_{(2,3)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

This allows us to write

$$\begin{aligned} W &= \{1, (\psi\varphi), (\psi\varphi)^2, \psi, (\psi\varphi)\psi, (\psi\varphi)^2\psi\} \\ &= \langle (\psi\varphi), \psi : (\psi\varphi)^3 = \psi^2 = 1, \psi(\psi\varphi)\psi^{-1} = (\psi\varphi)^{-1} \rangle. \end{aligned}$$

Then, for each of the $w \in W$, we find the following:

| w | permutation | weight space $\Phi(w)$ |
|---|---|---|
| $(\psi\varphi)$ | (1,2,3) | (1,3)(2,3) |
| $(\psi\varphi)^2$ | (1,3,2) | (1,2)(1,3) |
| $\psi$ | (1,2) | (1,2) |
| $(\psi\varphi)\psi$ | (1,3) | (1,2)(1,3)(2,3) |
| $(\psi\varphi)^2\psi$ | (2,3) | (2,3) |

**Theorem 4.4.1** *Let $k_\nu$ be a local field with valuation $\nu$. Then, for each $\alpha_3 \in T$ and with $\psi$, $\varphi \in W$ as defined above, the cocycle $dec_\nu$ satisfies,*

$$\begin{aligned} dec_\nu(\psi, \alpha_3) = dec_\nu(s_{(1,2)}, \alpha_3) &= dec_\nu(s_{(1,2)}, \alpha_2^{(1,2)}) \\ &= (-1)^{\frac{(\rho-1)}{2^r}\max\{X_1,X_2\}} \end{aligned}$$

$$\begin{aligned} dec_\nu(\varphi, \alpha_3) = dec_\nu(s_{(2,3)}, \alpha_3) &= dec_\nu(s_{(1,2)}, \alpha_2^{(2,3)}) \\ &= (-1)^{\frac{(\rho-1)}{2^r}\max\{X_2,X_3\}}, \end{aligned}$$

*where $(-1) = 1$ whenever $m = t$ is odd and where we define $\alpha_2^{(i,j)} \in GL_2(k_\nu)$ to be the diagonal matrix, $\alpha_2^{(i,j)} = \mathrm{diag}(\pi^{X_i}a_i, \pi^{X_j}a_j)$.*

**PROOF OF THEOREM:**

We shall begin by calculating,

$$\begin{aligned} dec_\nu(\psi,\alpha_3) &= \prod_{\xi\in\mu_m} \xi^{\int_{k_\nu{}^3\setminus 0} f\left(\left(\begin{smallmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)\mathcal{X}\right) f(\xi\mathcal{X})\left(M(\mathcal{X})-M(\alpha_3^{-1}\mathcal{X})\right)\mathrm{d}\mathcal{X}} \\ &= \prod_{\xi\in\mu_m} \xi^{\int_{k_\nu{}^3\setminus 0} f\left(\begin{smallmatrix}\psi\hat{\mathcal{X}} \\ x_3\end{smallmatrix}\right) f\left(\begin{smallmatrix}\xi\hat{\mathcal{X}} \\ \xi x_3\end{smallmatrix}\right)\left(M(\mathcal{X})-M(\alpha_3^{-1}\mathcal{X})\right)\mathrm{d}\hat{\mathcal{X}}\mathrm{d}x_3}, \end{aligned}$$

where as previously defined we have $\hat{\mathcal{X}} = (x_1, x_2)^T \in k_\nu{}^2$.

Since we are hoping to build on the ideas used in section 4.3 we decompose $k_\nu{}^3\setminus 0$ into just two disjoint open sets,

$$\begin{aligned} A_1 &= \{(\hat{\mathcal{X}}, x_3)^T \in k_\nu{}^3 \setminus 0 : |\hat{\mathcal{X}}|_\nu \geq |x_3|_\nu\} \\ A_2 &= \{(\hat{\mathcal{X}}, x_3)^T \in k_\nu{}^3 \setminus 0 : |\hat{\mathcal{X}}|_\nu < |x_3|_\nu\}, \end{aligned}$$

where $|\hat{\mathcal{X}}|_\nu = |(x_1,x_2)^T|_\nu = \max\{|x_1|_\nu, |x_2|_\nu\}$ as in Section 2.

Let us now define $I_i^\psi(\xi)$ to be the integral over $A_i$ for each $i$. Then, for each $\xi \neq 1$, we find

$$\begin{aligned} I_1^\psi(\xi) &= \int_{A_1} f(\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\hat{\mathcal{X}}) f(\xi\hat{\mathcal{X}})(M(\mathcal{X}) - M(\alpha_3^{-1}\mathcal{X}))\mathrm{d}\hat{\mathcal{X}}\mathrm{d}x_3 \\ I_2^\psi(\xi) &= \int_{A_2} f(x_3) f(\xi x_3)(M(\mathcal{X}) - M(\alpha_3^{-1}\mathcal{X}))\mathrm{d}\mathcal{X} = 0. \end{aligned}$$

We now define the set $A_1(\hat{\mathcal{X}}) = \{z \in k_\nu : (\hat{\mathcal{X}}, z)^T \in A_1\}$. Then, using Lemma 1, we find that the integral $I_1^\psi(\xi)$ satisfies,

$$\begin{aligned} I_1^\psi(\xi) &= \int\limits_{k_\nu{}^2\setminus 0} f(\psi\hat{\mathcal{X}}) f(\xi\hat{\mathcal{X}}) \left\{ \left( M(\hat{\mathcal{X}}) \int\limits_{A_1(\hat{\mathcal{X}})} M(x_3)\mathrm{d}x_3 \right) - \left( M(\alpha_2^{-1}\hat{\mathcal{X}}) \int\limits_{A_1(\hat{\mathcal{X}})} M(\pi^{-X_3} a_3^{-1} x_3)\mathrm{d}x_3 \right) \right\} \mathrm{d}\hat{\mathcal{X}} \\ &\equiv \int_{k_\nu{}^2\setminus 0} f(\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\hat{\mathcal{X}}) f(\xi\hat{\mathcal{X}})(M(\hat{\mathcal{X}}) - M(\alpha_2^{-1}\hat{\mathcal{X}}))\mathrm{d}\hat{\mathcal{X}} \qquad (\mathrm{mod}\ m). \end{aligned}$$

However, by referring to Section 4.3, we see that this integral has already been calculated. That is,

$$I_1^\psi(\xi) \equiv I_2(\xi) \qquad (\mathrm{mod}\ m).$$

Therefore we may immediately conclude that,

$$\begin{aligned} dec_\nu(\psi,\alpha_3) = dec_\nu(s_{(1,2)},\alpha_3) &= dec_\nu(s_{(1,2)},\alpha_2^{(1,2)}) \\ &= (-1)^{\frac{(\rho-1)}{2^r}\max\{X_1,X_2\}}. \end{aligned}$$

Next we must consider the generator $\varphi$ of $W$. In this case we must now concentrate on calculating,

$$
\begin{aligned}
dec_\nu(\varphi,\alpha_3) &= \prod_{\xi\in\mu_m} \xi^{\int_{k_\nu{}^3\backslash 0} f\left(\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{smallmatrix}\right)\mathcal{X}\right) f(\xi\mathcal{X})\left(M(\mathcal{X})-M(\alpha_3^{-1}\mathcal{X})\right)\mathrm{d}\mathcal{X}} \\
&= \prod_{\xi\in\mu_m} \xi^{\int_{k_\nu{}^3\backslash 0} f\left(\begin{smallmatrix} x_1 \\ \psi\check{\mathcal{X}} \end{smallmatrix}\right) f\left(\begin{smallmatrix} \xi x_1 \\ \xi\check{\mathcal{X}} \end{smallmatrix}\right)\left(M(\mathcal{X})-M(\alpha_3^{-1}\mathcal{X})\right)\mathrm{d}x_1\mathrm{d}\check{\mathcal{X}}},
\end{aligned}
$$

where we have defined $\check{\mathcal{X}} = (x_2, x_3)^T$ with $|\check{\mathcal{X}}|_\nu = \max\{\,|x_2|_\nu, |x_3|_\nu\}$.

We once again decompose $k_\nu{}^3\backslash 0$ into two disjoint open sets,

$$
\begin{aligned}
A_1 &= \{(x_1,\check{\mathcal{X}})^T \in k_\nu{}^3 \setminus 0 : |x_1|_\nu \geq |\check{\mathcal{X}}|_\nu\} \\
A_2 &= \{(x_1,\check{\mathcal{X}})^T \in k_\nu{}^3 \setminus 0 : |x_1|_\nu < |\check{\mathcal{X}}|_\nu\}.
\end{aligned}
$$

As in the previous section we define $I_i^\varphi(\xi)$ be the integral over $A_i$ for each $i$. Then, for each $\xi \neq 1$, we find

$$
\begin{aligned}
I_1^\varphi(\xi) &= \int_{A_1} f(x_1)f(\xi x_1)\left(M(\mathcal{X}) - M(\alpha_3^{-1}\mathcal{X})\right)\mathrm{d}\mathcal{X} = 0 \\
I_2^\varphi(\xi) &= \int_{A_2} f\left(\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\check{\mathcal{X}}\right) f(\xi\check{\mathcal{X}})\left(M(\mathcal{X}) - M(\alpha_3^{-1}\mathcal{X})\right)\mathrm{d}x_1\mathrm{d}\check{\mathcal{X}}.
\end{aligned}
$$

Furthermore, by defining the set $A_2(\check{\mathcal{X}}) = \{x \in k_\nu : (x,\check{\mathcal{X}})^T \in A_2\}$ and once again using Lemma 1, we are able to express this last integral as,

$$
\begin{aligned}
I_2^\varphi(\xi) &= \int\limits_{k_\nu{}^2\backslash 0} f(\psi\check{\mathcal{X}})f(\xi\check{\mathcal{X}})\left\{\left(M(\check{\mathcal{X}})\int\limits_{A_2(\check{\mathcal{X}})} M(x_1)\mathrm{d}x_1\right) - \left(M((\alpha_2^{(2,3)})^{-1}\check{\mathcal{X}})\int\limits_{A_2(\check{\mathcal{X}})} M(\pi^{-X_1}a_1^{-1}x_1)\mathrm{d}x_1\right)\right\}\mathrm{d}\check{\mathcal{X}} \\
&\equiv \int\limits_{k_\nu{}^2\backslash 0} f\left(\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\check{\mathcal{X}}\right)f(\xi\check{\mathcal{X}})\left(M(\check{\mathcal{X}}) - M((\alpha_2^{(2,3)})^{-1}\check{\mathcal{X}})\right)\mathrm{d}\check{\mathcal{X}} \qquad (\mathrm{mod}\ m).
\end{aligned}
$$

Once again we realize that by replacing $\alpha_2$ with the matrix $\alpha_2^{(2,3)}$ this is precisely the integral we had calculated in Section 4.3. That is,

$$
I_2^\varphi(\xi) \equiv I_2(\xi) \qquad (\mathrm{mod}\ m).
$$

So finally we are able to conclude that the cocycle $dec_\nu$ satisfies,

$$
\begin{aligned}
dec_\nu(\varphi,\alpha_3) = dec_\nu(s_{(2,3)},\alpha_3) &= dec_\nu(s_{(1,2)},\alpha_2^{(2,3)}) \\
&= (-1)^{\frac{(\rho-1)}{2^r}\max\{X_2,X_3\}}. \qquad \square
\end{aligned}
$$

**Remark:**

By simply using the cocycle rule we discover that,

$$dec_\nu(w_1 w_2, \alpha_n) = dec_\nu(w_1, w_2\alpha_n)\, dec_\nu(w_2, \alpha_n)$$
$$= dec_\nu(w_1, {}^{w_2}\alpha_n)\, dec_\nu(w_2, \alpha_n).$$

for each $\alpha_n \in T$ and $w_1,\ w_2 \in W$.

Using this remark, for each $w \in W$ in $\mathrm{GL}_3(k_\nu)$, we calculate

$$\begin{aligned}
dec_\nu((\psi\varphi), \alpha_3) &= dec_\nu(\psi, {}^{\varphi}\alpha_3)\, dec_\nu(\varphi, \alpha_3) \\
&= (-1)^{\frac{(\rho-1)}{2^r}\max\{X_1,X_3\}}(-1)^{\frac{(\rho-1)}{2^r}\max\{X_2,X_3\}} \\
dec_\nu((\psi\varphi)^2, \alpha_3) &= dec_\nu((\psi\varphi), {}^{(\psi\varphi)}\alpha_3)\, dec_\nu((\psi\varphi), \alpha_3) \\
&= (-1)^{\frac{(\rho-1)}{2^r}\max\{X_1,X_2\}}(-1)^{\frac{(\rho-1)}{2^r}\max\{X_1,X_3\}} \\
dec_\nu(\psi, \alpha_3) &= dec_\nu(\psi, \alpha_3) \\
&= (-1)^{\frac{(\rho-1)}{2^r}\max\{X_1,X_2\}} \\
dec_\nu((\psi\varphi)\psi, \alpha_3) &= dec_\nu((\psi\varphi), {}^{\psi}\alpha_3)\, dec_\nu(\psi, \alpha_3) \\
&= (-1)^{\frac{(\rho-1)}{2^r}\max\{X_1,X_2\}}(-1)^{\frac{(\rho-1)}{2^r}\max\{X_1,X_3\}}(-1)^{\frac{(\rho-1)}{2^r}\max\{X_2,X_3\}} \\
dec_\nu((\psi\varphi)^2\psi, \alpha_3) &= dec_\nu((\psi\varphi)^2, {}^{\psi}\alpha_3)\, dec_\nu(\psi, \alpha_3) \\
&= (-1)^{\frac{(\rho-1)}{2^r}\max\{X_2,X_3\}}.
\end{aligned}$$

By considering the table on page 81 we see that we may express these results in the following theorem.

**Theorem 4.4.2** *For each $\alpha_3 \in T$ and $w \in W$ we have,*

$$dec_\nu(w, \alpha_3) = \prod_{(i,j)\in\Phi(w)} (-1)^{\frac{(\rho-1)}{2^r}\max\{X_i,X_j\}}.$$

*By using the cocycle rule we may conclude that on the monomials $M \subset GL_3(k_\nu)$ the 2-cocycle $dec_\nu$ satisfies,*

$$\begin{aligned}
dec_\nu(\alpha_3 w_1, \beta_3 w_2) &= dec_\nu(\alpha_3, {}^{w_1}\beta_3)\, dec_\nu(w_1, \beta_3) \\
&= \prod_{(i,j)\in\Phi(w_1)} (-1)^{\frac{(\rho-1)}{2^r}\max\{Y_i,Y_j\}}\, dec_\nu(\alpha_3, {}^{w_1}\beta_3),
\end{aligned}$$

*for each $w_1, w_2 \in W$.*

## 4.5 The cocycle $dec_\nu$ on $M \subset \mathrm{GL}_n(k_\nu)$

**Theorem 4.5.1** *Let $k_\nu$ be a local field with valuation $\nu$. Then for each $\alpha_n, \beta_n \in T$ and $w \in W$ we find,*

$$dec_\nu(w, \alpha_n) = \prod_{(i,j)\in\Phi(w)} (-1)^{\frac{(\rho-1)}{2^r}\max\{X_i,X_j\}},$$

*where $(-1) = 1$ whenever $m = t$ is odd.*

*Hence on the full group of monomials $M \subset GL_n(k_\nu)$ the cocycle $dec_\nu$ satisfies,*

$$\begin{aligned} dec_\nu(\alpha_n w_1, \beta_n w_2) &= dec_\nu(\alpha_n, {}^{w_1}\beta_n)\, dec_\nu(w_1, \beta_n) \\ &= \prod_{(i,j)\in\Phi(w_1)} (-1)^{\frac{(\rho-1)}{2^r}\max\{Y_i,Y_j\}}\, dec_\nu(\alpha_n, {}^{w_1}\beta_n). \end{aligned}$$

*for each $w_1, w_2 \in W$.*

We know that the group of permutation matrices $W$ is generated by the simple reflections. So in order to prove this theorem we shall first need to consider the value of $dec_\nu(w, \alpha_n)$ where $w = s_{(i,i+1)}$ is one such simple reflection. Before we begin let us define some new notation.

**Definition:**

For each root $\varsigma = (i,j) \in \Phi$ we define,

$$\widehat{\varsigma(X)} := \max\{X_i, X_j\}.$$

**Lemma 4.5.1** *For each $\alpha_n \in T$ and $w = s_{(i,i+1)} \in W$ the cocycle $dec_\nu$ satisfies,*

$$\begin{aligned} dec_\nu(w, \alpha_n) = dec_\nu(s_{(i,i+1)}, \alpha_n) &= dec_\nu(s_{(1,2)}, \alpha_2^{(i,i+1)}) \\ &= (-1)^{\frac{(\rho-1)}{2^r}\max\{X_i,X_{i+1}\}} =: (-1)^{\frac{(\rho-1)}{2^r}\widehat{\varsigma(X)}}, \end{aligned}$$

*where $\varsigma = (i, i+1) \in \Phi$ and $(-1) = 1$ whenever $m = t$ is odd.*

**PROOF:**

In order to prove this lemma we shall expand on the ideas used in Lemma 4.4.1. By definition we must calculate,

$$dec_\nu(w, \alpha_n) = \prod_{\xi\in\mu_m} \xi^{\int_{k_\nu{}^n\setminus 0} f(s_{(i,i+1)}\mathcal{X}) f(\xi\mathcal{X})\left(M(\mathcal{X})-M(\alpha_n^{-1}\mathcal{X})\right)\mathrm{d}\mathcal{X}}.$$

Instead of decomposing $k_\nu{}^n\backslash 0$ into two sets, as we did on page 81, we must now use three. However this will require some new notation.

Let us define,

$$\begin{aligned}
\mid \hat{\mathcal{X}} \mid_\nu = \;& |(x_1, \dots, x_{i-1})^T|_\nu &&= \max\{\mid x_1 \mid_\nu, \dots, |x_{i-1}|_\nu\} \\
\mid \bar{\mathcal{X}} \mid_\nu = \;& |(x_i, x_{i+1})^T|_\nu &&= \max\{\mid x_i \mid_\nu, |x_{i+1}|_\nu\} \\
\mid \check{\mathcal{X}} \mid_\nu = \;& |(x_{i+2}, \dots, x_n)^T|_\nu &&= \max\{|x_{i+2}|_\nu, \dots, \mid x_n \mid_\nu\}.
\end{aligned}$$

Then we may decompose $k_\nu{}^n\backslash 0$ into the three disjoint open sets,

$$\begin{aligned}
A_1 &= \{\mathcal{X} \in k_\nu{}^n \setminus 0 : |\hat{\mathcal{X}}|_\nu \geq |\bar{\mathcal{X}}|_\nu,\ |\hat{\mathcal{X}}|_\nu \geq |\check{\mathcal{X}}|_\nu\} \\
A_2 &= \{\mathcal{X} \in k_\nu{}^n \setminus 0 : |\check{\mathcal{X}}|_\nu > |\hat{\mathcal{X}}|_\nu,\ |\check{\mathcal{X}}|_\nu > |\bar{\mathcal{X}}|_\nu\} \\
A_3 &= \{\mathcal{X} \in k_\nu{}^n \setminus 0 : |\bar{\mathcal{X}}|_\nu > |\hat{\mathcal{X}}|_\nu,\ |\bar{\mathcal{X}}|_\nu \geq |\check{\mathcal{X}}|_\nu\}.
\end{aligned}$$

Defining $I_i^w(\xi)$ to be the integral over $A_i$ for each $i$ we immediately find that, as in the proof of Theorem 4.3.1, for $\xi \neq 1$ our integrals satisfy,

$$\begin{aligned}
I_1^w(\xi) &= \int_{A_1} f(\hat{\mathcal{X}}) f(\xi\hat{\mathcal{X}}) \big(M(\mathcal{X}) - M(\alpha_n^{-1}\mathcal{X})\big) \mathrm{d}\mathcal{X} = 0 \\
I_2^w(\xi) &= \int_{A_2} f(\check{\mathcal{X}}) f(\xi\check{\mathcal{X}}) \big(M(\mathcal{X}) - M(\alpha_n^{-1}\mathcal{X})\big) \mathrm{d}\mathcal{X} = 0.
\end{aligned}$$

So we need only consider the integral $I_3^w(\xi)$ where,

$$\begin{aligned}
I_3^w(\xi) &= \int_{A_3} f\big(\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\bar{\mathcal{X}}\big) f(\xi\bar{\mathcal{X}}) \big(M(\mathcal{X}) - M(\alpha_n^{-1}\mathcal{X})\big) \mathrm{d}\mathcal{X} \\
&= \int_{A_3} f\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_i \\ x_{i+1} \end{pmatrix}\right) f\left(\xi\begin{pmatrix} x_i \\ x_{i+1} \end{pmatrix}\right) \big(M(\mathcal{X}) - M(\alpha_n^{-1}\mathcal{X})\big) \mathrm{d}\mathcal{X}.
\end{aligned}$$

In order to calculate this integral we must first define the set,

$$A_3(\bar{\mathcal{X}}) = \{(\hat{\mathcal{X}}, \check{\mathcal{X}})^T \in k_\nu{}^{n-2} \setminus 0 : (\hat{\mathcal{X}}, \bar{\mathcal{X}}, \check{\mathcal{X}})^T \in A_3\}.$$

Then, by expressing the diagonal matrix $\alpha_n \in T$ as,

$$\alpha_n = \mathrm{diag}(\alpha_{\hat{\mathcal{X}}}, \alpha_2^{(i,i+1)}, \alpha_{\check{\mathcal{X}}}),$$

and using Lemma 1 we are able to write,

$$
\begin{aligned}
I_3^w(\xi) = \int_{k_\nu{}^2\backslash 0} & f(s_{(1,2)}\bar{\mathcal{X}})f(\xi\bar{\mathcal{X}}) \\
& \left\{\left(M(\bar{\mathcal{X}}) \int_{A_3(\bar{\mathcal{X}})} M(\hat{\mathcal{X}})M(\check{\mathcal{X}})\mathrm{d}\hat{\mathcal{X}}\mathrm{d}\check{\mathcal{X}}\right)\right. \\
& \qquad \left. - \left(M((\alpha_2^{(i,i+1)})^{-1}\bar{\mathcal{X}}) \int_{A_3(\bar{\mathcal{X}})} M(\alpha_{\check{\mathcal{X}}}^{-1}\hat{\mathcal{X}})M(\alpha_{\check{\mathcal{X}}}^{-1}\check{\mathcal{X}})\,\mathrm{d}\hat{\mathcal{X}}\mathrm{d}\check{\mathcal{X}}\right)\right\}\mathrm{d}\bar{\mathcal{X}} \\
= \int_{k_\nu{}^2\backslash 0} & f(s_{(1,2)}\bar{\mathcal{X}})f(\xi\bar{\mathcal{X}}) \\
& \left\{\left(M(\bar{\mathcal{X}}) \int_{A_3(\bar{\mathcal{X}})} M((\hat{\mathcal{X}},\check{\mathcal{X}})^T)\,\mathrm{d}(\hat{\mathcal{X}},\check{\mathcal{X}})\right)\right. \\
& \qquad \left. - \left(M((\alpha_2^{(i,i+1)})^{-1}\bar{\mathcal{X}}) \int_{A_3(\bar{\mathcal{X}})} M(\alpha_{\hat{\mathcal{X}}\oplus\check{\mathcal{X}}}^{-1}(\hat{\mathcal{X}},\check{\mathcal{X}})^T)\,\mathrm{d}(\hat{\mathcal{X}},\check{\mathcal{X}})\right)\right\}\mathrm{d}\bar{\mathcal{X}} \\
\equiv \int_{k_\nu{}^2\backslash 0} & f(\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\bar{\mathcal{X}})f(\xi\bar{\mathcal{X}})(M(\bar{\mathcal{X}}) - M((\alpha_2^{(i,i+1)})^{-1}\bar{\mathcal{X}}))\mathrm{d}\bar{\mathcal{X}} \qquad (\mathrm{mod}\ m),
\end{aligned}
$$

which is once again identical to the integral we had calculated in Section 4.3.

Therefore, letting $\varsigma = (i, i+1)$ be any simple root, we are able to conclude that the cocycle satisfies,

$$
\begin{aligned}
dec_\nu(w,\alpha_n) = dec_\nu(s_{(i,i+1)},\alpha_n) &= dec_\nu(s_{(1,2)},\alpha_2^{(i,i+1)}) \\
&= (-1)^{\frac{(p-1)}{2^r}\max\{X_i,X_{i+1}\}} \\
&:= (-1)^{\frac{(p-1)}{2^r}\widehat{\varsigma(X)}}.
\end{aligned}
$$

□

**Proof of Theorem 4.5.1:**

Before we begin the proof of the main theorem in this section we let,

$$w = s_{\varsigma_1}\dots s_{\varsigma_\ell}, \quad \text{with } \varsigma_k = (i_k, i_k+1) \in \Delta,$$

be some reduced expression for $w$ such that $l(w) = \ell$.

We have already seen that the cocycle $dec_\nu$ satisfies,

$$
\begin{aligned}
dec_\nu(w_1w_2,\alpha_n) &= dec_\nu(w_1, w_2\alpha_n)\,dec_\nu(w_2,\alpha_n) \\
&= dec_\nu(w_1, {}^{w_2}\alpha_n)\,dec_\nu(w_2,\alpha_n).
\end{aligned}
$$

Therefore, by repeatedly applying the cocycle rule, we are able to calculate,

$$
\begin{aligned}
dec_\nu(w, \alpha_n) &= dec_\nu(s_{\varsigma_1} \dots s_{\varsigma_\ell}, \alpha_n) \\
&= dec_\nu(s_{\varsigma_1} \dots s_{\varsigma_{\ell-1}}, {}^{s_{\varsigma_\ell}} \alpha_n)\, dec_\nu(s_{\varsigma_\ell}, \alpha_n) \\
&= dec_\nu(s_{\varsigma_1} \dots s_{\varsigma_{\ell-2}}, {}^{s_{\varsigma_{\ell-1}} s_{\varsigma_\ell}} \alpha_n)\, dec_\nu(s_{\varsigma_{\ell-1}}, {}^{s_{\varsigma_\ell}} \alpha_n)\, dec_\nu(s_{\varsigma_\ell}, \alpha_n)
\end{aligned}
$$

and continuing in this way we eventually find,

$$
\begin{aligned}
&= dec_\nu(s_{\varsigma_1}, {}^{s_{\varsigma_2} \dots s_{\varsigma_\ell}} \alpha_n) \dots dec_\nu(s_{\varsigma_{\ell-1}}, {}^{s_{\varsigma_\ell}} \alpha_n)\, dec_\nu(s_{\varsigma_\ell}, \alpha_n) \\
&= \prod_{k=1}^{\ell} dec_\nu(s_{\varsigma_k}, {}^{s_{\varsigma_{k+1}} \dots s_{\varsigma_\ell}} \alpha_n) \\
&= \prod_{k=1}^{\ell} dec_\nu(s_{\varsigma_k}, \alpha_n^{s_{\varsigma_\ell} \dots s_{\varsigma_{k+1}}}). \qquad (4.2)
\end{aligned}
$$

However, by Lemma 4.5.1, on the simple reflections the cocycle $dec_\nu$ is known to satisfy,

$$dec_\nu(s_{\varsigma_k}, \alpha_n) = (-1)^{\frac{(\rho-1)}{2^r} \max\{X_{i_k}, X_{i_k+1}\}} := (-1)^{\frac{(\rho-1)}{2^r} \widehat{\varsigma_k(X)}}.$$

Substituting this into the equation 4.2 we find,

$$
\begin{aligned}
dec_\nu(w, \alpha_n) &= \prod_{k=1}^{\ell} (-1)^{\frac{(\rho-1)}{2^r} \widehat{s_{\varsigma_\ell} s_{\varsigma_{\ell-1}} \dots s_{\varsigma_{k+1}} (\varsigma_k)(X)}} \\
&=: \prod_{\varsigma \in \Lambda} (-1)^{\frac{(\rho-1)}{2^r} \widehat{\varsigma(X)}},
\end{aligned}
$$

where, by Lemma 4.1.1, the set $\Lambda$ is given by,

$$\Lambda = \{ s_{\varsigma_\ell} \dots s_{\varsigma_2}(\varsigma_1), \dots, s_{\varsigma_\ell}(\varsigma_{\ell-1}), \varsigma_\ell \} = \Phi(w).$$

Therefore, in conclusion we have proved that the cocycle $dec_\nu$ satisfies,

$$
\begin{aligned}
dec_\nu(w, \alpha_n) &= \prod_{\varsigma \in \Phi(w)} (-1)^{\frac{(\rho-1)}{2^r} \widehat{\varsigma(X)}} \\
&= \prod_{(i,j) \in \Phi(w)} (-1)^{\frac{(\rho-1)}{2^r} \max\{X_i, X_j\}}.
\end{aligned}
$$

We are now able to turn our attention towards on the full group of monomials $M \subset \mathrm{GL}_n(k_\nu)$. Using the cocycle rule we discover that, for each $w_1, w_2 \in W$ and $\alpha_n, \beta_n \in T$, the cocycle $dec_\nu$ satisfies,

$$
\begin{aligned}
dec_\nu(\alpha_n w_1, \beta_n w_2) &= dec_\nu(\alpha_n, {}^{w_1}\beta_n)\, dec_\nu(w_1, \beta_n) \\
&= dec_\nu(\alpha_n, {}^{w_1}\beta_n). \prod_{(i,j) \in \Phi(w_1)} (-1)^{\frac{(\rho-1)}{2^r} \max\{Y_i, Y_j\}},
\end{aligned}
$$

which concludes the proof of our main theorem. □

## 4.6 The cocycle $dec_\nu$ on the monomials $M = T.\mathfrak{M}$

In the latter parts of this thesis we shall be looking at the splitting of the quotient of the two cocycles $dec_\nu$ and $\sigma_n$. However, as we have seen the cocycle $\sigma_n$ is defined on the monomials in a slightly different way to that which we have described above. Since $\sigma_n$ is obtained by restriction of a cocycle on $\mathrm{SL}_{n+1}$ we can no longer consider the group of permutation matrices $W$ since these may have determinant $\pm 1$. As described in the introduction we instead consider a complete set of coset representatives for $W$. That is, we consider the set

$$\mathfrak{M} = \{\eta_w : w \in W\}$$

where, for each $w \in W$, we define

$$\eta_w = w_{\varsigma_1} \dots w_{\varsigma_\ell},$$

and $w = s_{\varsigma_1} \dots s_{\varsigma_\ell}$ is some reduced expression for $w$. Let us also recall that the map $w \mapsto \eta_w$ is independent of this representation of $w$ as a minimal product of simple reflections.

Since we clearly have,

$$M = \coprod_{\eta_w \in \mathfrak{M}} T\eta_w,$$

for the remainder of this chapter we shall consider how the cocycle $dec_\nu$ behaves on the space $T \times \mathfrak{M}$ instead of $T \times W$.

**Remark:**

For each generator $s_\varsigma \in W$ with $\varsigma = (i, i+1)$ we have,

$$\eta_{s_\varsigma} = w_\varsigma = \epsilon_\varsigma s_\varsigma \in \mathfrak{M},$$

where $\epsilon_\varsigma = \epsilon_i$ is the diagonal matrix with a $-1$ in the $i^{th}$ position and ones elsewhere. Therefore, if $w = s_{\varsigma_1} \dots s_{\varsigma_\ell} \in W$ is some reduced expression for $w$ we must have,

$$\begin{aligned}
\eta_w &= w_{\varsigma_1} \dots w_{\varsigma_\ell} \\
&= \epsilon_{\varsigma_1} s_{\varsigma_1} \dots \epsilon_{\varsigma_\ell} s_{\varsigma_\ell} \\
&= \epsilon_{\varsigma_1} {}^{s_{\varsigma_1}}\epsilon_{\varsigma_2} \dots {}^{s_{\varsigma_1} \dots s_{\varsigma_{\ell-1}}} \epsilon_{\varsigma_\ell} \,.\, s_{\varsigma_1} \dots s_{\varsigma_\ell} \\
&= \epsilon_{\varsigma_1} \epsilon_{\varsigma_2}^{s_{\varsigma_1}} \dots \epsilon_{\varsigma_\ell}^{s_{\varsigma_1} \dots s_{\varsigma_{\ell-1}}} w \\
&=: \epsilon_w w,
\end{aligned}$$

where the diagonal matrix $\epsilon_w$ is defined to be,

$$\epsilon_w := \epsilon_{\varsigma_1} \epsilon_{\varsigma_2}^{s_{\varsigma_1}} \dots \epsilon_{\varsigma_\ell}^{s_{\varsigma_1} \dots s_{\varsigma_{\ell-1}}}.$$

**Theorem 4.6.1** *Let $k_\nu$ be a local field with valuation $\nu$. Then for each $g \in GL_n(k_\nu)$ and each $\eta_w \in \mathfrak{M}$ the cocycle $dec_\nu$ satisfies,*

$$dec_\nu(g, \eta_w) = 1. \tag{4.3}$$

**PROOF OF THEOREM:**

Once again, since we have $\mathfrak{M} \in \mathrm{GL}_n(\mathfrak{O}_\nu)$, the proof of this statement follows from Theorem 1.4.1 on page 26. □

**Corollary 2** *By applying the cocycle rule we are able to further deduce that for each $g_1, g_2 \in GL_n(k_\nu)$ and each $\eta_w \in \mathfrak{M}$ the cocycle $dec_\nu$ satisfies,*

$$dec_\nu(g_1, g_2\eta_w) = dec_\nu(g_1, g_2).$$

**Theorem 4.6.2** *Let $k_\nu$ be a local field with valuation $\nu$. Then for each $\alpha_n, \beta_n \in T$ and $\eta_w \in \mathfrak{M}$ the cocycle $dec_\nu$ satisfies,*

$$dec_\nu(\eta_w, \alpha_n) = \prod_{(i,j)\in\Phi(w)} (-1)^{\frac{(\rho-1)}{2^r}X_j}(-1)^{\frac{(\rho-1)}{2^r}\max\{X_i,X_j\}},$$

*where $(-1) = 1$ whenever $m = t$ is odd.*

*Hence, on the full group of monomials $M \subset GL_n(k_\nu)$ the cocycle $dec_\nu$ satisfies,*

$$dec_\nu(\alpha_n\eta_{w_1}, \beta_n\eta_{w_2}) = \prod_{(i,j)\in\Phi(w_1)} (-1)^{\frac{(\rho-1)}{2^r}Y_j}(-1)^{\frac{(\rho-1)}{2^r}\max\{Y_i,Y_j\}}\, dec_\nu(\alpha_n, {}^{w_1}\beta_n).$$

**PROOF OF THEOREM:**

By considering the previous remark and applying the cocycle rule we find,

$$\begin{aligned} dec_\nu(\eta_w, \alpha_n) &= dec_\nu(\epsilon_w w, \alpha_n) \\ &= dec_\nu(\epsilon_w, {}^w\alpha_n)\, dec_\nu(w, \alpha_n). \end{aligned}$$

Therefore, using the result given in Theorem 4.5.1, in order to prove this statement it shall be sufficient for us to show that,

$$dec_\nu(\epsilon_w, {}^w\alpha_n) = \prod_{(i,j)\in\Phi(w)} (-1)^{\frac{(\rho-1)}{m}X_j}.$$

However, using Theorem 2.4.1 from Chapter 2, we already know that on the torus $T$ the cocycle $dec_\nu$ satisfies,

$$dec_\nu(\epsilon_i, \alpha_n) = dec_\nu(-1, \pi^{X_i}a_i) = (-1)^{\frac{(\rho-1)}{2^r}X_i}.$$

Let $w = s_{\varsigma_1} \dots s_{\varsigma_\ell} \in W$ be some reduced expression for $w$ and let us also define $\eta_w := \epsilon_w w$ as in the Remark on page 89. Furthermore, if we also let $\epsilon_{\varsigma_k} = \epsilon_{i_k}$ for each $\varsigma_k = (i_k, i_k+1) \in \Phi$, then by repeatedly applying the cocycle rule we find,

$$\begin{aligned}
dec_\nu(\epsilon_w, {}^w\alpha_n) &= dec_\nu(\epsilon_{\varsigma_1}\, \epsilon_{\varsigma_2}^{s_{\varsigma_1}} \dots \epsilon_{\varsigma_\ell}^{s_{\varsigma_1}\dots s_{\varsigma_{\ell-1}}},\ {}^{s_{\varsigma_1}\dots s_{\varsigma_\ell}}\alpha_n) \\
&= \prod_{k=1}^{\ell} dec_\nu(\epsilon_{\varsigma_k}^{s_{\varsigma_1}\dots s_{\varsigma_{k-1}}}, \alpha_n^{s_{\varsigma_\ell}\dots s_{\varsigma_1}}) = \prod_{k=1}^{\ell} dec_\nu(\epsilon_{s_{\varsigma_1}\dots s_{\varsigma_{k-1}}(i_k)}, \alpha_n^{s_{\varsigma_\ell}\dots s_{\varsigma_1}}) \\
&= \prod_{k=1}^{\ell} (-1)^{\frac{(\rho-1)}{2^r} X_{s_{\varsigma_\ell}\dots s_{\varsigma_1}\cdot s_{\varsigma_1}\dots s_{\varsigma_{k-1}}(i_k)}} = \prod_{k=1}^{\ell} (-1)^{\frac{(\rho-1)}{2^r} X_{s_{\varsigma_\ell}\dots s_{\varsigma_k}(i_k)}} \\
&= \prod_{k=1}^{\ell} (-1)^{\frac{(\rho-1)}{2^r} X_{s_{\varsigma_\ell}\dots s_{\varsigma_{k+1}}(i_k+1)}} = \prod_{j\in\Lambda} (-1)^{\frac{(\rho-1)}{2^r} X_j}.
\end{aligned}$$

However, by Lemma 4.1.1, we know that

$$\begin{aligned}
\Phi(w) &= \{\varsigma_\ell, s_{\varsigma_\ell}(\varsigma_{\ell-1}), s_{\varsigma_\ell}s_{\varsigma_{\ell-1}}(\varsigma_{\ell-2}), \dots, s_{\varsigma_\ell}\dots s_{\varsigma_2}(\varsigma_1)\} \\
&= \{(i_\ell, i_\ell+1), s_{\varsigma_\ell}(i_{\ell-1}, i_{\ell-1}+1), s_{\varsigma_\ell}s_{\varsigma_{\ell-1}}(i_{\ell-2}, i_{\ell-2}+1), \dots, s_{\varsigma_\ell}\dots s_{\varsigma_2}(i_1, i_1+1)\} \\
&= \cup_{k=1}^{\ell}\{(s_{\varsigma_\ell}\dots s_{\varsigma_{k+1}}(i_k), s_{\varsigma_\ell}\dots s_{\varsigma_{k+1}}(i_k+1))\}.
\end{aligned}$$

After careful consideration we notice that the set $\Lambda$ of subscripts is in fact the set of $j$'s for which $(i,j)$ is contained in the weight space $\Phi(w)$. That is,

$$dec_\nu(\epsilon_w, {}^w\alpha_n) = \prod_{j\in\Lambda} (-1)^{\frac{(\rho-1)}{2^r} X_j} = \prod_{(i,j)\in\Phi(w)} (-1)^{\frac{(\rho-1)}{2^r} X_j}.$$

Having obtained this result, by simply applying the cocycle rule, on the full group of monomials $M \subset \mathrm{GL}_n(k_\nu)$ we find,

$$\begin{aligned}
dec_\nu(\alpha_n\eta_{w_1}, \beta_n\eta_{w_2}) &= dec_\nu(\alpha_n, {}^{w_1}\beta_n)\, dec_\nu(\eta_{w_1}, \beta_n) \\
&= \prod_{(i,j)\in\Phi(w_1)} (-1)^{\frac{(\rho-1)}{2^r} Y_j} (-1)^{\frac{(\rho-1)}{2^r}\max\{Y_i, Y_j\}}\, dec_\nu(\alpha_n, {}^{w_1}\beta_n),
\end{aligned}$$

as required.

□

## 4.7 The coboundary $\partial\tau_n$

To complete this chapter we shall once again consider the coboundary $\partial\tau_n$ associated with $dec_\nu$ on the torus in the case that $m$, the number of roots of unity in $\mu_m$, is even. This final theorem shall be of importance when we come to consider the splitting of the quotient of cocycles $dec_\nu$ and $\sigma_n$ in the later chapters.

**Theorem 4.7.1** *Let $k_\nu$ be a local field with valuation $\nu$. Then for each $\alpha_n \in T$ in $GL_n(k_\nu)$ and $w \in W$ the function $\tau_n$ satisfies,*

$$\tau_n({}^w\alpha_n) = \tau_n(\alpha_n) \cdot \prod_{\varsigma\in\Phi(w)} (-1)^{\frac{(\rho-1)}{2^r}X_iX_j}(-1)^{\frac{(\rho-1)}{2^r}\max(X_i,X_j)}. \tag{4.4}$$

**PROOF OF THEOREM:**

Let us recall that in the previous chapter we had defined the function $\tau_n$ by,

$$\begin{aligned}\tau_n(\alpha_n) &= \jmath_2(\alpha_2)\jmath_3(\alpha_3)\dots\jmath_n(\alpha_n)\\ &= \prod_{(i,j)\in\Phi^+} (-1)^{\frac{(\rho-1)}{2^r}X_i\max(X_i,X_j)}(-1)^{\frac{(\rho-1)}{2^r}\frac{\max(X_i,X_j)(\max(X_i,X_j)+1)}{2}}.\end{aligned}$$

Since we have an action of $W$ on the set $\Phi$ of roots and since we have $\max(X_j, X_i) = \max(X_i, X_j)$ we are able to write,

$$\begin{aligned}\tau_n(\alpha_n^w) &= \prod_{(i,j)\in\Phi^+} (-1)^{\frac{(\rho-1)}{2^r}X_{w(i)}\max(X_{w(i)},X_{w(j)})}(-1)^{\frac{(\rho-1)}{2^r}\frac{\max(X_{w(i)},X_{w(j)})(\max(X_{w(i)},X_{w(j)})+1)}{2}}\\ &= \prod_{(i,j)\in\Phi^+} (-1)^{\frac{(\rho-1)}{2^r}X_{w(i)}\max(X_{w(i)},X_{w(j)})}(-1)^{\frac{(\rho-1)}{2^r}\frac{\max(X_i,X_j)(\max(X_i,X_j)+1)}{2}}.\end{aligned}$$

Therefore, considering the quotient we find that,

$$\frac{\tau_n({}^w\alpha_n)}{\tau_n(\alpha_n)} = \prod_{(i,j)\in\Phi^+} (-1)^{\frac{(\rho-1)}{2^r}X_i\max(X_i,X_j)}. \prod_{(i,j)\in\Phi^+} (-1)^{\frac{(\rho-1)}{2^r}X_{w^{-1}(i)}\max(X_{w^{-1}(i)},X_{w^{-1}(j)})}.$$

By simply relabeling this becomes,

$$= \prod_{(i,j)\in\Phi^+} (-1)^{\frac{(\rho-1)}{2^r}X_i\max(X_i,X_j)}. \prod_{(l,m)\in\Phi^+} (-1)^{\frac{(\rho-1)}{2^r}X_{w^{-1}(l)}\max(X_{w^{-1}(l)},X_{w^{-1}(m)})}.$$

If we now let $w(i) = l$ and $w(j) = m$ we are further able to write,

$$\frac{\tau_n({}^{w}\alpha_n)}{\tau_n(\alpha_n)} = \prod_{(i,j)\in\Phi^+} (-1)^{\frac{(\rho-1)}{2^r} X_i \max(X_i,X_j)} \cdot \prod_{(w(i),w(j))\in\Phi^+} (-1)^{\frac{(\rho-1)}{2^r} X_i \max(X_i,X_j)}$$

Splitting the last product in this expression this becomes,

$$= \prod_{(i,j)\in\Phi^+} (-1)^{\frac{(\rho-1)}{2^r} X_i \max(X_i,X_j)} \cdot \prod_{\substack{(w(i),w(j))\in\Phi^+,\\(i,j)\in\Phi^+}} \cdot \prod_{\substack{(w(i),w(j))\in\Phi^+,\\(i,j)\in\Phi^-}} (-1)^{\frac{(\rho-1)}{2^r} X_i \max(X_i,X_j)}$$

$$= \prod_{(i,j)\in\Phi^+} (-1)^{\frac{(\rho-1)}{2^r} X_i \max(X_i,X_j)} \cdot \prod_{(i,j)\in\Phi^+\setminus\Phi(w)} \cdot \prod_{(j,i)\in\Phi(w)} (-1)^{\frac{(\rho-1)}{2^r} X_i \max(X_i,X_j)}$$

By simply canceling we are left with,

$$= \prod_{(i,j)\in\Phi(w)} (-1)^{\frac{(\rho-1)}{2^r} X_i \max(X_i,X_j)} \cdot \prod_{(j,i)\in\Phi(w)} (-1)^{\frac{(\rho-1)}{2^r} X_i \max(X_i,X_j)}$$

$$= \prod_{\varsigma\in\Phi(w)} (-1)^{\frac{(\rho-1)}{2^r} (X_i+X_j) \max(X_i,X_j)}.$$

Finally, carrying out a case by case analysis this is found to be,

$$\frac{\tau_n({}^{w}\alpha_n)}{\tau_n(\alpha_n)} = \prod_{\varsigma\in\Phi(w)} (-1)^{\frac{(\rho-1)}{2^r} X_i X_j} (-1)^{\frac{(\rho-1)}{2^r} \max(X_i,X_j)},$$

which completes the proof of our theorem.

□

# Chapter 5

# Calculation of $dec_\nu$ on $\mathrm{GL}_2(k_\nu)$

## 5.1 Introduction

Once again we begin by letting $k_\nu$ be a local field with valuation $\nu$ and a fixed uniformizing element $\pi$ in $\mathfrak{O}_\nu$. Let us recall that having defined $\mathfrak{M} = \{\eta_w : w \in W\}$, the map $w \mapsto \eta_w$ gives a bijection $W \tilde{\to} \mathfrak{M}$ and $\mathfrak{M}$ is a complete set of coset representatives for $W \cong \mathbb{M}/\mathbb{T}$.

Using this, for $\mathrm{GL}_n(k_\nu)$, we have the Bruhat decomposition:

$$\mathrm{GL}_n(k_\nu) = \coprod_{\eta_w \in \mathfrak{M}} NT\eta_w N,$$

and there exists a unique map $\mathsf{t} : \mathrm{GL}_n(k_\nu) \to T$ such that

$$\mathsf{t}(nt\eta_w n') = t, \quad \forall n,\ n' \in N,\ t \in T,\ \eta_w \in \mathfrak{M}.$$

where $\mathsf{t}(g)$ may be easily computed for any $g \in \mathrm{GL}_n(k_\nu)$.

Let us also recall that in $\mathrm{GL}_2(k_\nu)$ the group $N$ is simply generated by,

$$N = \left\langle n_{1,2}(\pi^{-Z}c) = \begin{pmatrix} 1 & \pi^{-Z}c \\ 0 & 1 \end{pmatrix} : \pi^{-Z}c \in k_\nu \right\rangle.$$

As with the previous chapters we define the diagonal matrix $\alpha_2 := \mathrm{diag}(\pi^{X_1}a_1, \pi^{X_2}a_2)$.

Throughout this chapter we shall be using this Bruhat decomposition and finding some more results for the cocycle $dec_\nu$. In particular we shall be considering $dec_\nu$ on the following spaces:

$$T \times N, \qquad TW \times N, \qquad NT \times N, \quad \text{and} \quad NTW \times N.$$

Although these may seem fairly spurious there is a good reason why we concentrate on these particular results. Using the Bruhat decomposition for $\mathrm{GL}_2$ these results will allow us, in the later chapters, to completely describe the splitting of the quotient of Hill's and Matsumoto's cocycles. We will therefore, in the case of $GL_2$, be able to express the isomorphism between the corresponding metaplectic groups explicitly.

## 5.2 $dec_\nu$ on $GL_2(k_\nu) \times N$

**Theorem 5.2.1** *Let $k_\nu$ be a local field with valuation $\nu$. Then, for any matrix $g$ and for each $n_{1,2}(\pi^{-Z}c) \in N$ with $Z \leq 0$, the cocycle $dec_\nu$ satisfies,*

$$dec_\nu(g, n_{1,2}(\pi^{-Z}c)) = 1.$$

**PROOF OF THEOREM:**

Since we have $Z \leq 0$ we find that $n_{1,2}(\pi^{-Z}c) \in GL_2(\mathfrak{O}_\nu)$. Using this fact our result follows by Theorem 1.4.1. □

### 5.2.1 Our Method of Calculation for $dec_\nu(g, n_2)$

For the remainder of this chapter we shall define,

$$n_2 := n_{1,2}(\pi^{-Z_2}c_2) \in N \qquad \text{with} \quad Z_2 > 0.$$

By definition, for each $g \in GL_2(k_\nu)$, we know that the cocycle satisfies,

$$\begin{aligned} dec_\nu(g, n_{1,2}(\pi^{-Z_2}c_2)) &= \prod_{\xi \in \mu_m} \xi^{\int_{k_\nu^2} f(g\mathcal{X}) f(\xi\mathcal{X}) \left(M(\mathcal{X}) - M(n_2^{-1}\mathcal{X})\right) d\mathcal{X}} \\ &= \prod_{\xi \in \mu_m} \xi^{\left\{\int_{\mathfrak{O}_\nu^2} - \int_{n_2 \mathfrak{O}_\nu^2}\right\} f(g\mathcal{X}) f(\xi\mathcal{X}) d\mathcal{X}} =: \prod_{\xi \in \mu_m} \xi^{I(\xi)}. \end{aligned}$$

However, since $Z_2 > 0$ this is no longer trivial. Therefore, we shall need to find some way of calculating the integral $I(\xi)$.

Firstly, we shall consider the space over which we are integrating. Having defined the set $\Pi$ to be a set of representatives for $\mathfrak{O}_\nu/\pi\mathfrak{O}_\nu$ including 0 we may dissect the space $\mathfrak{O}_\nu^2$ into disjoint sets $A(i_o, \ldots, i_{Z_2-1})$ defined such that,

$$\begin{aligned} \mathfrak{O}_\nu^2 &= \bigcup_{i_{Z_2-1} \in \Pi} \cdots \bigcup_{i_o \in \Pi} \left\{ \begin{pmatrix} x_1 \\ \pi^{Z_2}x_2 + \pi^{Z_2-1}i_{Z_2-1} + \ldots + \pi i_1 + i_o \end{pmatrix} : x_1, x_2 \in \mathfrak{O}_\nu \right\} \\ &:= \bigcup_{i_{Z_2-1} \in \Pi} \cdots \bigcup_{i_o \in \Pi} A(i_o, \ldots, i_{Z_2-1}). \end{aligned}$$

If we now define,

$$I_1^{(\xi)}(i_o,\dots,i_{Z_2-1}) = \int_{A(i_o,\dots,i_{Z_2-1})} f(g\mathcal{X})f(\xi\mathcal{X})\mathrm{d}\mathcal{X}$$

$$\text{and}\qquad I_2^{(\xi)}(i_o,\dots,i_{Z_2-1}) = \int_{n_2A(i_o,\dots,i_{Z_2-1})} f(g\mathcal{X})f(\xi\mathcal{X})\mathrm{d}\mathcal{X},$$

then, using this dissection, we may write

$$\begin{aligned} I(\xi) &= \sum_{i_{Z_2-1}\in\Pi}\cdots\sum_{i_o\in\Pi}\left\{\int_{A(i_o,\dots,i_{Z_2-1})} - \int_{n_2A(i_o,\dots,i_{Z_2-1})}\right\} f(g\mathcal{X})f(\xi\mathcal{X})\mathrm{d}\mathcal{X} \\ &= \sum_{i_{Z_2-1}\in\Pi}\cdots\sum_{i_o\in\Pi}\left(I_1^{(\xi)}(i_o,\dots,i_{Z_2-1}) - I_2^{(\xi)}(i_o,\dots,i_{Z_2-1})\right). \end{aligned}$$

**Lemma 5.2.1**

$$I_1^{(\xi)}(0,\dots,0) - I_2^{(\xi)}(0,\dots,0) = 0.$$

**PROOF:**

By simply noticing that,

$$\begin{aligned} n_2A(0,\dots,0) &= \left\{\begin{pmatrix} 1 & \pi^{-Z_2}c_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ \pi^{Z_2}x_2 \end{pmatrix} : x_1, x_2 \in \mathfrak{O}_\nu\right\} \\ &= \left\{\begin{pmatrix} x_1 + c_2x_2 \\ \pi^{Z_2}x_2 \end{pmatrix} : x_1, x_2 \in \mathfrak{O}_\nu\right\} \end{aligned}$$

which, having made the simple change of variable $x_1 + c_2x_2 \longmapsto x_1$, is equal to

$$= \left\{\begin{pmatrix} x_1 \\ \pi^{Z_2}x_2 \end{pmatrix} : x_1, x_2 \in \mathfrak{O}_\nu\right\} = A(0,\dots,0),$$

our result quickly follows. □

Using the previous lemma we may now write,

$$\begin{aligned} I(\xi) &= \sum_{i_{Z_2-1}\in\Pi}\cdots\sum_{i_o\in\Pi} I_1^{(\xi)}(i_o,\dots,i_{Z_2-1}) - \sum_{i_{Z_2-1}\in\Pi}\cdots\sum_{i_o\in\Pi} I_2^{(\xi)}(i_o,\dots,i_{Z_2-1}) \\ &= \sum_{k=0}^{Z_2-1}\left(\sum_{i_{Z_2-1}\in\Pi}\cdots\sum_{i_{k+1}\in\Pi}\ \sum_{i_k\in\Pi\backslash 0} I_1^{(\xi)}(0,\dots,0,i_k,\dots,i_{Z_2-1})\right) \\ &\qquad - \sum_{k=0}^{Z_2-1}\left(\sum_{i_{Z_2-1}\in\Pi}\cdots\sum_{i_{k+1}\in\Pi}\ \sum_{i_k\in\Pi\backslash 0} I_2^{(\xi)}(0,\dots,0,i_k,\dots,i_{Z_2-1})\right). \end{aligned}$$

Now, in the theorems which follow we find that the integrals $I_\ell^{(\xi)}(0,.,0,i_k,.,i_{Z-1})$ depend only on the value of $i_k$. Therefore, in order to make the notation simpler, we shall immediately define

$$I_1^{(\xi)}(i_k) := I_1^{(\xi)}(0,.,0,i_k,.,i_{Z_2-1}) \quad \text{and} \quad I_2^{(\xi)}(i_k) := I_2^{(\xi)}(0,.,0,i_k,.,i_{Z_2-1}),$$

being careful to remember that we are in fact considering a set of similar integrals.

The reason for choosing this dissection should now become clear as finally, using this new notation, we are further able to deduce that:

$$\begin{aligned} I(\xi) &= \sum_{k=0}^{Z_2-1}\Big(\sum_{i_{Z_2-1}\in\Pi}\cdots\sum_{i_{k+1}\in\Pi}\sum_{i_k\in\Pi\backslash 0} I_1^{(\xi)}(i_k)\Big) - \sum_{k=0}^{Z_2-1}\Big(\sum_{i_{Z_2-1}\in\Pi}\cdots\sum_{i_{k+1}\in\Pi}\sum_{i_k\in\Pi\backslash 0} I_2^{(\xi)}(i_k)\Big) \\ &\equiv \sum_{k=0}^{Z_2-1}\sum_{i_k\in\Pi\backslash 0} I_1^{(\xi)}(i_k) - \sum_{k=0}^{Z_2-1}\sum_{i_k\in\Pi\backslash 0} I_2^{(\xi)}(i_k) \pmod m. \end{aligned}$$

Therefore, when calculating the value of $dec_\nu(g,n_2)$, we shall in fact concentrate on calculating the integrals $I_1^{(\xi)}(i_k)$ and $I_2^{(\xi)}(i_k)$ with $i_k$ strictly non-zero. Once we have these results we will then be able to reconstruct the expression,

$$\begin{aligned} dec_\nu(g,n_{1,2}(\pi^{-Z_2}c_2)) &= \prod_{\xi\in\mu_m}\xi^{I(\xi)} \\ &= \prod_{\xi\in\mu_m}\xi^{\sum_{k=0}^{Z_2-1}\sum_{i_k\in\Pi\backslash 0} I_1^{(\xi)}(i_k)}\prod_{\xi\in\mu_m}\xi^{-\sum_{k=0}^{Z_2-1}\sum_{i_k\in\Pi\backslash 0} I_2^{(\xi)}(i_k)}, \end{aligned}$$

and use the following lemmas to evaluate the cocycle.

**Lemma 5.2.2** *Suppose the expression $J^{(\xi)}(i_k) \in \mathbb{Z}/m$ is independent of $i_k$. Then we find,*

$$\begin{aligned} \prod_{\xi\in\mu_m}\xi^{\sum_{k=0}^{Z_2-1}\sum_{i_k\in\Pi\backslash 0} J^{(\xi)}(i_k)} &= \prod_{k=0}^{Z_2-1}\prod_{\xi\in\mu_m}\left(\xi^{\sum_{i_k\in\Pi\backslash 0}1}\right)^{J^{(\xi)}(i_k)} \\ &= \prod_{k=0}^{Z_2-1}\prod_{\xi\in\mu_m}\left(\xi^{(\rho-1)}\right)^{J^{(\xi)}(i_k)} = 1, \end{aligned}$$

*is trivial since $(\rho-1)\equiv 0 \pmod m$.*

**Lemma 5.2.3** *Suppose that the expression $J^{(\xi)}(i_k) \in \mathbb{Z}/m$ is independent of $\xi$. Then we find,*

$$\prod_{\xi \in \mu_m} \xi^{\sum_{k=0}^{Z_2-1} \sum_{i_k \in \Pi \backslash 0} J^{(\xi)}(i_k)} = \prod_{k=0}^{Z_2-1} \Big( \prod_{\xi \in \mu_m} \xi \Big)^{\sum_{i_k \in \Pi \backslash 0} J^{(\xi)}(i_k)}$$
$$= \prod_{k=0}^{Z_2-1} (-1)^{\sum_{i_k \in \Pi \backslash 0} J^{(\xi)}(i_k)},$$

*where, as we have seen, this is trivial whenever $m$ is odd. However, when $m$ is even, we see that in order to calculate this we need only consider the expression $J^{(\xi)}(i_k)$ modulo 2.*

Finally, before we begin, let us recall the simplified form of the Gauss-Schering Lemma that we first saw in Chapter 2. For the purpose of this chapter it is more convenient for us to state this lemma in the following way.

**Lemma 5.2.4** *Having fixed a uniformizing element $\pi$ we have,*

$$\prod_{\xi \in \mu_m} \xi^{\sum_{i_k \in \Pi \backslash 0} f(a i_k) f(\xi i_k)} = (a, \pi)_{\nu, m},$$

*where $\Pi$ is as previously described.*

### Important Remark:

Throughout this chapter, when calculating the cocycle, we shall repeatedly use these lemmas. That is, when calculating $I_1^{(\xi)}(i_k)$ and $I_2^{(\xi)}(i_k)$, we shall simply ignore terms independent of $i_k$ and only consider the terms independent of $\xi$ modulo 2. We shall also use our simplified version of the Gauss-Schering Lemma without further reference.

To this end, with regard to these integrals, we shall employ a serious abuse of notation. We shall use the equals sign to express the fact that two expressions are equal as exponents of the product of roots, even when the expressions themselves may not be equal.

## 5.3 $dec_\nu$ on $T \times N$

**Theorem 5.3.1** *For each $\alpha_2 \in T$ and each $n_2 := n_{1,2}(\pi^{-Z_2}c_2) \in N$ with $Z_2 > 0$ the cocycle $dec_\nu$ satisfies,*

- $X_1 - X_2 \le 0 < Z_2$ :

$$dec_\nu\left(\begin{pmatrix} \pi^{X_1}a_1 & 0 \\ 0 & \pi^{X_2}a_2 \end{pmatrix}, \begin{pmatrix} 1 & \pi^{-Z_2}c_2 \\ 0 & 1 \end{pmatrix}\right) = (a_2/a_1, \pi)_{\nu,m}^{Z_2} (-1)^{\frac{(\rho-1)}{2^r} Z_2(X_1-X_2)}$$

- $0 \le X_1 - X_2 \le Z_2$ :

$$dec_\nu\left(\begin{pmatrix} \pi^{X_1}a_1 & 0 \\ 0 & \pi^{X_2}a_2 \end{pmatrix}, \begin{pmatrix} 1 & \pi^{-Z_2}c_2 \\ 0 & 1 \end{pmatrix}\right) = (a_2/a_1, \pi)_{\nu,m}^{Z_2} (-1)^{\frac{(\rho-1)}{2^r} Z_2(X_1-X_2)} (-1)^{\frac{(\rho-1)}{2^r}\frac{(X_1-X_2)(X_1-X_2-1)}{2}}$$

- $0 < Z_2 < X_1 - X_2$ :

$$dec_\nu\left(\begin{pmatrix} \pi^{X_1}a_1 & 0 \\ 0 & \pi^{X_2}a_2 \end{pmatrix}, \begin{pmatrix} 1 & \pi^{-Z_2}c_2 \\ 0 & 1 \end{pmatrix}\right) = (c_2, \pi)_{\nu,m}^{Z_2} (-1)^{\frac{(\rho-1)}{2^r}\frac{Z_2(Z_2+1)}{2}}.$$

**PROOF OF THEOREM:**

As we saw in the previous section 5.2.1, in order to evaluate the cocycle we must first consider the integrals $I_1^{(\xi)}(i_k)$ and $I_2^{(\xi)}(i_k)$.

### The Integrals $I_1^{(\xi)}(i_k)$

Having made the substitutions,

$$x_2 \longmapsto \pi^{Z_2}x_2 + \pi^{Z_2-1}i_{Z_2-1} + \ldots + \pi^k i_k \quad \Rightarrow \mathrm{d}x_2 \longmapsto |\pi^{Z_2}|_\nu \mathrm{d}x_2 \equiv \mathrm{d}x_2 \pmod m,$$

for the integral $I_1^{(\xi)}(i_k)$ we are able to write,

$$\begin{aligned} I_1^{(\xi)}(i_k) &= \int_{\mathfrak{O}_\nu^2} f\left(\tfrac{\pi^{X_1}a_1x_1}{\pi^{X_2}a_2(\pi^{Z_2}x_2+\ldots+\pi^k i_k)}\right) f\left(\tfrac{\xi x_1}{\xi(\pi^{Z_2}x_2+\ldots+\pi^k i_k)}\right)\mathrm{d}\mathcal{X} \\ &= \int_{\mathfrak{O}_\nu^2} f\left(\tfrac{\pi^{X_1}a_1x_1}{\pi^{X_2+k}a_2i_k}\right) f\left(\tfrac{\xi x_1}{\xi\pi^k i_k}\right)\mathrm{d}\mathcal{X} \\ &= \int_{\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu\setminus\pi^{k+1}\mathfrak{O}_\nu} f\left(\tfrac{\pi^{X_1}a_1x_1}{\pi^{X_2+k}a_2i_k}\right) f(\xi x_1)\mathrm{d}\mathcal{X} + f(\xi i_k)\int_{\mathfrak{O}_\nu}\int_{\pi^{k+1}\mathfrak{O}_\nu} f\left(\tfrac{\pi^{X_1}a_1x_1}{\pi^{X_2+k}a_2i_k}\right)\mathrm{d}\mathcal{X}, \end{aligned}$$

where, as we had previously stated, this integral depends only on the value of $i_k$.

To solve these integrals shall require us to consider three separate cases,

$$\text{(i) } 0 \le k < X_1 - X_2, \quad \text{(ii) } 0 \le X_1 - X_2 \le k, \quad \text{(iii) } X_1 - X_2 \le 0 \le k.$$

**Case (i)** : $0 \leq k < X_1 - X_2$

$$I_1^{(\xi)}(i_k) = f(a_2 i_k) \int_{\mathfrak{O}_\nu} \int_{\mathfrak{O}_\nu \setminus \pi^{k+1}\mathfrak{O}_\nu} f(\xi x_1) \mathrm{d}\mathcal{X} + f(a_2 i_k) f(\xi i_k) \int_{\mathfrak{O}_\nu} \int_{\pi^{k+1}\mathfrak{O}_\nu} \mathrm{d}\mathcal{X}$$

However, since $f$ is a fundamental function, this is simply

$$= \frac{1}{m}(\rho^{k+1} - 1) f(a_2 i_k) + f(a_2 i_k) f(\xi i_k).$$

So, taking the sum over the $i_k$'s, we find that

$$\begin{aligned}
\sum_{i_k \in \Pi \setminus 0} I_1^{(\xi)}(i_k) &= \frac{1}{m}(\rho^{k+1} - 1) \sum_{i_k} f(a_2 i_k) + \sum_{i_k \in \Pi \setminus 0} f(a_2 i_k) f(\xi i_k) \\
&= \frac{1}{m}(\rho^{k+1} - 1) \frac{(\rho - 1)}{m} + \sum_{i_k \in \Pi \setminus 0} f(a_2 i_k) f(\xi i_k) \\
&= \frac{(\rho - 1)}{m}(k + 1) + \sum_{i_k \in \Pi \setminus 0} f(a_2 i_k) f(\xi i_k),
\end{aligned}$$

where, as in Lemma 5.2.3, we simply considered the first term modulo 2.

**Case (ii)** : $0 \leq X_1 - X_2 \leq k$ :

$$\begin{aligned}
I_1^{(\xi)}(i_k) = \int\limits_{\mathfrak{O}_\nu} \int\limits_{\mathfrak{O}_\nu \setminus \pi^{k-(X_1-X_2)+1}\mathfrak{O}_\nu} f(a_1 x_1) f(\xi x_1) \, \mathrm{d}\mathcal{X} \; + \; f(a_2 i_k) \int\limits_{\mathfrak{O}_\nu} \int\limits_{\pi^{k-(X_1-X_2)+1}\mathfrak{O}_\nu \setminus \pi^{k+1}\mathfrak{O}_\nu} f(\xi x_1) \; \mathrm{d}\mathcal{X} \\
+ f(a_2 i_k) f(\xi i_k) \int_{\mathfrak{O}_\nu} \int_{\pi^{k+1}\mathfrak{O}_\nu} \mathrm{d}\mathcal{X}
\end{aligned}$$

Disregarding the first term (by Lemma 5.2.2) and using the fact that $f$ is fundamental this is simply,

$$= \frac{1}{m}(\rho^{(X_1 - X_2)} - 1) f(a_2 i_k) + f(a_2 i_k) f(\xi i_k).$$

Using Lemma 5.2.3 and performing a similar calculation to that in Case (i), when summing over the $i_k$'s, we find

$$\sum_{i_k \in \Pi \setminus 0} I_1^{(\xi)}(i_k) = \frac{(\rho - 1)}{m}(X_1 - X_2) + \sum_{i_k \in \Pi \setminus 0} f(a_2 i_k) f(\xi i_k).$$

**Case (iii) : $X_1 - X_2 \leq 0 \leq k$ :**

$$I_1^{(\xi)}(i_k) = \int\limits_{\mathfrak{O}_\nu} \int\limits_{\mathfrak{O}_\nu \backslash \pi^{k+1}\mathfrak{O}_\nu} f(a_1 x_1) f(\xi x_1) \mathrm{d}\mathcal{X} \; + \; f(\xi i_k) \int\limits_{\mathfrak{O}_\nu} \int\limits_{\pi^{k+1}\mathfrak{O}_\nu \backslash \pi^{k-(X_1-X_2)+1}\mathfrak{O}_\nu} f(a_1 x_1)\, \mathrm{d}\mathcal{X}$$
$$+ f(a_2 i_k) f(\xi i_k) \int_{\mathfrak{O}_\nu} \int_{\pi^{k-(X_1-X_2)+1}\mathfrak{O}_\nu} \mathrm{d}\mathcal{X}$$

One again, disregarding the first term and noting that $f$ is fundamental, we have

$$= \frac{1}{m}(\rho^{(X_2-X_1)} - 1) f(\xi i_k) + f(a_2 i_k) f(\xi i_k).$$

Summing over the $i_k$'s, in this case we find

$$\sum_{i_k \in \Pi\backslash 0} I_1^{(\xi)}(i_k) = \frac{(\rho - 1)}{m}(X_2 - X_1) + \sum_{i_k \in \Pi\backslash 0} f(a_2 i_k) f(\xi i_k).$$

We should point out that, since the first term is only considered modulo 2, Cases (ii) and (iii) are identical. However, for the purposes of this proof we shall leave these results as two distinct cases.

**The Integrals $I_2^{(\xi)}(i_k)$**

In order to calculate the integrals $I_2^{(\xi)}(i_k)$ we shall require the substitutions,

$$x_1 \longmapsto x_1 + \pi^{-Z_2} c_2 (\pi^{Z_2} x_2 + \pi^{Z_2-1} i_{Z_2-1} + \ldots + \pi^k i_k)$$
$$x_2 \longmapsto \pi^{Z_2} x_2 + \pi^{Z_2-1} i_{Z_2-1} + \ldots + \pi^k i_k$$
$$\Rightarrow \quad \mathrm{d}\mathcal{X} \longmapsto \rho^{-Z_2} \mathrm{d}\mathcal{X} \equiv \mathrm{d}\mathcal{X} \pmod{m}.$$

Using these substitutions we are able to calculate,

$$I_2^{(\xi)}(i_k) = \int_{\mathfrak{O}_\nu^2} f\genfrac(){0pt}{}{\pi^{X_1} a_1 (x_1 + \pi^{-Z_2} c (\pi^{Z_2} x_2 + \ldots + \pi^k i_k))}{\pi^{X_2} a_2 (\pi^{Z_2} x_2 + \ldots + \pi^k i_k)} f\genfrac(){0pt}{}{\xi (x_1 + \pi^{-Z_2} c_2 (\pi^{Z_2} x_2 + \ldots + \pi^k i_k))}{\xi (\pi^{Z_2} x_2 + \ldots + \pi^k i_k)} \mathrm{d}\mathcal{X}$$
$$= f(\xi c_2 i_k) \int_{\mathfrak{O}_\nu^2} f\genfrac(){0pt}{}{\pi^{X_1-(Z_2-k)} a_1 c_2 i_k}{\pi^{X_2+k} a_2 i_k} \mathrm{d}\mathcal{X}.$$

Clearly, in order to solve this integral we shall require just two cases.

**Case (1) : $X_1 - X_2 \leq Z_2$**

Since we have $X_1 - (Z_2 - k) \leq X_2 + k$ we simply find,

$$\sum_{i_k \in \Pi\backslash 0} I_2^{(\xi)}(i_k) = \sum_{i_k \in \Pi\backslash 0} f(a_1 c_2 i_k) f(\xi c_2 i_k).$$

**Case (2) : $X_1 - X_2 > Z_2$ :**

In this case we have $X_1 - (Z_2 - k) > X_2 + k$. Therefore we find,

$$\sum_{i_k \in \Pi \backslash 0} I_2^{(\xi)}(i_k) = \sum_{i_k \in \Pi \backslash 0} f(a_2 i_k) f(\xi c_2 i_k).$$

Since, for each $0 \leq k < Z_2$, we have calculated the integrals $I_1^{(\xi)}(i_k)$ and $I_2^{(\xi)}(i_k)$ and found that these integrals depend only on the value of $i_k$ we do indeed find,

$$I(\xi) \equiv \sum_{k=0}^{Z_2-1} \sum_{i_k \in \Pi \backslash 0} I_1^{(\xi)}(i_k) - \sum_{k=0}^{Z_2-1} \sum_{i_k \in \Pi \backslash 0} I_2^{(\xi)}(i_k) \pmod{m}.$$

Finally, in order to prove the theorem it simply remains to consider the various cases which can arise. Since this is the first time we are presenting these calculations we shall be careful to explain our method in detail.

For each of the three cases we present a table showing which results we require for the integrals $I_1^{(\xi)}(i_k)$, $I_2^{(\xi)}(i_k)$ for any given $k$. Then, compiling these results together with lemmas 5.2.2, 5.2.3 and 2.0.5, we are finally able to calculate the cocycle.

**Results for $X_1 - X_2 \leq 0 < Z_2$ :**

| | $0 \leq k < Z_2$ |
|---|---|
| $I_1^{(\xi)}(i_k)$ | Case (iii) |
| $I_2^{(\xi)}(i_k)$ | Case (1) |

Using the results found in Case (iii) and Case (1) we find,

$$\begin{aligned}
dec_\nu(\alpha_2, n_2) &= \prod_{k=0}^{Z_2-1} \prod_{\xi \in \mu_m} \xi^{\sum_{i_k} I_1^{(\xi)}(i_k)} \prod_{k=0}^{Z_2-1} \prod_{\xi \in \mu_m} \xi^{-\sum_{i_k} I_2^{(\xi)}(i_k)} \\
&= \prod_{k=0}^{Z_2-1} \Big( \prod_{\xi \in \mu_m} \xi \Big)^{\frac{(\rho-1)}{m}(X_2 - X_1)} \prod_{k=0}^{Z_2-1} \prod_{\xi \in \mu_m} \xi^{\sum_{i_k} f(a_2 i_k) f(\xi i_k)} \prod_{k=0}^{Z_2-1} \prod_{\xi \in \mu_m} \xi^{-\sum_{i_k} f(a_1 c_2 i_k) f(\xi c_2 i_k)} \\
&= \prod_{k=0}^{Z_2-1} (-1)^{\frac{(\rho-1)}{m}(X_2 - X_1)} \prod_{k=0}^{Z_2-1} (a_2, \pi)_{\nu,m} \prod_{k=0}^{Z_2-1} (a_1, \pi)_{\nu,m}^{-1} \\
&= (a_2/a_1, \pi)_{\nu,m}^{Z_2} \quad (-1)^{\frac{(\rho-1)}{2^r} Z_2 (X_2 - X_1)}.
\end{aligned}$$

**Results for** $0 \le X_1 - X_2 \le Z_2$ **:**

| | $0 \le k < X_1 - X_2$ | $X_1 - X_2 \le k < Z_2$ |
|---|---|---|
| $I_1^{(\xi)}(i_k)$ | Case (i) | Case (ii) |
| $I_2^{(\xi)}(i_k)$ | Case (1) | |

In this case, depending on the value of $k$, we shall have two possible results for the integrals $I_1^{(\xi)}(i_k)$. Therefore, we shall split the product over $k$ of the integrals $I_1^{(\xi)}(i_k)$ accordingly.

$$
\begin{aligned}
dec_\nu(\alpha_2, n_2) &= \prod_{k=0}^{X_1-X_2-1} \prod_{\xi\in\mu_m} \xi^{\sum_{i_k} I_1^{(\xi)}(i_k)} \prod_{k=X_1-X_2}^{Z_2-1} \prod_{\xi\in\mu_m} \xi^{\sum_{i_k} I_1^{(\xi)}(i_k)} \prod_{k=0}^{Z_2-1} \prod_{\xi\in\mu_m} \xi^{-\sum_{i_k} I_2^{(\xi)}(i_k)} \\
&= (\prod_{\xi\in\mu_m} \xi)^{\frac{(\rho-1)}{m}\sum_{k=0}^{X_1-X_2-1}(k+1)} \prod_{k=0}^{X_1-X_2-1} \prod_{\xi\in\mu_m} \xi^{\sum_{i_k} f(a_2 i_k) f(\xi i_k)} \\
&\quad \prod_{k=X_1-X_2}^{Z_2-1} (\prod_{\xi\in\mu_m} \xi)^{\frac{(\rho-1)}{m}(X_1-X_2)} \prod_{k=X_1-X_2}^{Z_2-1} \prod_{\xi\in\mu_m} \xi^{\sum_{i_k} f(a_2 i_k) f(\xi i_k)} \\
&\quad \prod_{k=0}^{Z_2-1} \prod_{\xi\in\mu_m} \xi^{-\sum_{i_k} f(a_1 c_2 i_k) f(\xi c_2 i_k)} \\
&= (-1)^{\frac{(\rho-1)}{m}\frac{(X_1-X_2)(X_1-X_2+1)}{2}} (a_2, \pi)_{\nu,m}^{X_1-X_2} \\
&\quad (-1)^{\frac{(\rho-1)}{m}(X_1-X_2)(Z_2-(X_1-X_2))} (a_2, \pi)_{\nu,m}^{Z_2-(X_1-X_2)} (a_1, \pi)_{\nu,m}^{-Z_2} \\
&= (a_2/a_1, \pi)_{\nu,m}^{Z_2} \quad (-1)^{\frac{(\rho-1)}{2^r} Z_2(X_1-X_2)} (-1)^{\frac{(\rho-1)}{2^r}\frac{(X_1-X_2)(X_1-X_2-1)}{2}}.
\end{aligned}
$$

**Results for** $0 < Z_2 < X_1 - X_2$ **:**

| | $0 \le k < Z_2$ |
|---|---|
| $I_1^{(\xi)}(i_k)$ | Case (i) |
| $I_2^{(\xi)}(i_k)$ | Case (2) |

In this final case we simply find,

$$
\begin{aligned}
dec_\nu(\alpha_2, n_2) &= \prod_{k=0}^{Z_2-1} \prod_{\xi\in\mu_m} \xi^{\sum_{i_k} I_1^{(\xi)}(i_k)} \quad \prod_{k=0}^{Z_2-1} \prod_{\xi\in\mu_m} \xi^{-\sum_{i_k} I_2^{(\xi)}(i_k)} \\
&= (\prod_{\xi\in\mu_m} \xi)^{\frac{(\rho-1)}{m}\sum_{k=0}^{Z_2-1}(k+1)} \prod_{k=0}^{Z_2-1} \prod_{\xi\in\mu_m} \xi^{\sum_{i_k} f(a_2 i_k) f(\xi i_k)} \prod_{k=0}^{Z_2-1} \prod_{\xi\in\mu_m} \xi^{-\sum_{i_k} f(a_2 i_k) f(\xi c_2 i_k)} \\
&= (-1)^{\frac{(\rho-1)}{m}\frac{Z_2(Z_2+1)}{2}} (a_2, \pi)_{\nu,m}^{Z_2} (a_2/c_2, \pi)_{\nu,m}^{-Z_2} \\
&= (c_2, \pi)_{\nu,m}^{Z_2} \quad (-1)^{\frac{(\rho-1)}{2^r}\frac{Z_2(Z_2+1)}{2}},
\end{aligned}
$$

which completes the proof of our theorem. □

## 5.4 $dec_\nu$ on $M \times N$

**Theorem 5.4.1** *Let $\eta_\psi \in \mathfrak{M}$ where $\psi = s_{(1,2)} \in W$ is as previously defined in chapter 4. Then, for each $\alpha_2 \in T$ and each $n_2 := n_{1,2}(\pi^{-Z_2}c_2) \in N$ with $Z_2 > 0$, the cocycle $dec_\nu$ satisfies,*

- $X_2 - X_1 < 0 < Z_2$:

$$dec_\nu\left(\left(\begin{smallmatrix}\pi^{X_1}a_1 & 0\\ 0 & \pi^{X_2}a_2\end{smallmatrix}\right)\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right), \left(\begin{smallmatrix}1 & \pi^{-Z_2}c_2\\ 0 & 1\end{smallmatrix}\right)\right) = (a_1/a_2, \pi)^{Z_2}_{\nu,m}\,(-1)^{\frac{(\rho-1)}{2^r}Z_2(X_2-X_1)}$$

- $0 \le X_2 - X_1 < Z_2$:

$$dec_\nu\left(\left(\begin{smallmatrix}\pi^{X_1}a_1 & 0\\ 0 & \pi^{X_2}a_2\end{smallmatrix}\right)\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right), \left(\begin{smallmatrix}1 & \pi^{-Z_2}c_2\\ 0 & 1\end{smallmatrix}\right)\right) = (a_1/a_2, \pi)^{Z_2}_{\nu,m}\,(-1)^{\frac{(\rho-1)}{2^r}Z_2(X_2-X_1)}\,(-1)^{\frac{(\rho-1)}{2^r}\frac{(X_2-X_1)(X_2-X_1+1)}{2}}$$

- $0 < Z_2 \le X_2 - X_1$:

$$dec_\nu\left(\left(\begin{smallmatrix}\pi^{X_1}a_1 & 0\\ 0 & \pi^{X_2}a_2\end{smallmatrix}\right)\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right), \left(\begin{smallmatrix}1 & \pi^{-Z_2}c_2\\ 0 & 1\end{smallmatrix}\right)\right) = (c_2, \pi)^{Z_2}_{\nu,m}\,(-1)^{\frac{(\rho-1)}{2^r}\frac{Z_2(Z_2+1)}{2}}.$$

**PROOF OF THEOREM:**

**The Integrals $I_1^{(\xi)}(i_k)$**

Having substituted,

$$x_2 \longmapsto \pi^{Z_2}x_2 + \pi^{Z_2-1}i_{Z_2-1} + \ldots + \pi^k i_k \quad \Rightarrow \mathrm{d}x_2 \longmapsto |\pi^{Z_2}|_\nu \mathrm{d}x_2 \equiv \mathrm{d}x_2 \pmod{m},$$

we may express the integral $I_1^{(\xi)}(i_k)$ as,

$$\begin{aligned}
I_1^{(\xi)}(i_k) &= \int_{\mathfrak{O}_\nu^2} f\left(\tfrac{-\pi^{X_1}a_1(\pi^{Z_2}x_2+\ldots+\pi^k i_k)}{\pi^{X_2}a_2x_1}\right) f\left(\tfrac{\xi x_1}{\xi(\pi^{Z_2}x_2+\ldots+\pi^k i_k)}\right)\mathrm{d}\mathcal{X}\\
&= \int_{\mathfrak{O}_\nu^2} f\left(\tfrac{-\pi^{X_1+k}a_1 i_k}{\pi^{X_2}a_2x_1}\right) f\left(\tfrac{\xi x_1}{\xi\pi^k i_k}\right)\mathrm{d}\mathcal{X}\\
&= f(\xi i_k)\int_{\mathfrak{O}_\nu}\int_{\pi^{k+1}\mathfrak{O}_\nu} f\left(\tfrac{-\pi^{X_1+k}a_1 i_k}{\pi^{X_2}a_2x_1}\right)\mathrm{d}\mathcal{X} + \int_{\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu\setminus\pi^{k+1}\mathfrak{O}_\nu} f\left(\tfrac{-\pi^{X_1+k}a_1 i_k}{\pi^{X_2}a_2x_1}\right) f(\xi x_1)\mathrm{d}\mathcal{X}.
\end{aligned}$$

As with the previous theorem, we shall again need to consider three separate cases.

$$\text{(i) } 0 \le k \le X_2 - X_1, \quad \text{(ii) } -1 \le X_2 - X_1 < k, \quad \text{(iii) } X_2 - X_1 < -1 < k.$$

We should point out here that the "$-1$" in these cases comes from the way in which we have defined the function $f$ (section 1.4.3) and the fact that we have switched our co-ordinates $x_1$ and $x_2$.

**Case (i)** : $0 \leq k \leq X_2 - X_1$ :

$$I_1^{(\xi)}(i_k) = f(-a_1 i_k) \int_{\mathfrak{O}_\nu} \int_{\mathfrak{O}_\nu \setminus \pi^{k+1}\mathfrak{O}_\nu} f(\xi x_1) \mathrm{d}\mathcal{X} + f(-a_1 i_k) f(\xi i_k) \int_{\mathfrak{O}_\nu} \int_{\pi^{k+1}\mathfrak{O}_\nu} \mathrm{d}\mathcal{X}$$
$$= \frac{1}{m}(\rho^{k+1} - 1) f(-a_1 i_k) + f(-a_1 i_k) f(\xi i_k).$$

So, taking the sum over the $i_k$'s, we find

$$\sum_{i_k \in \Pi \setminus 0} I_1^{(\xi)}(i_k) = \frac{1}{m}(\rho^{k+1} - 1) \sum_{i_k \in \Pi \setminus 0} f(-a_1 i_k) + \sum_{i_k \in \Pi \setminus 0} f(-a_1 i_k) f(\xi i_k)$$
$$= \frac{1}{m}(\rho^{k+1} - 1) \frac{(\rho - 1)}{m} + \sum_{i_k \in \Pi \setminus 0} f(-a_1 i_k) f(\xi i_k)$$
$$= \frac{(\rho - 1)}{m}(k+1) + \sum_{i_k \in \Pi \setminus 0} f(-a_1 i_k) f(\xi i_k),$$

where, by Lemma 5.2.3, the first term is only required modulo 2.

**Case (ii)** : $-1 \leq X_2 - X_1 < k$ :

$$I_1^{(\xi)}(i_k) = \int\limits_{\mathfrak{O}_\nu} \int\limits_{\mathfrak{O}_\nu \setminus \pi^{k-(X_2-X_1)}\mathfrak{O}_\nu} f(a_2 x_1) f(\xi x_1) \mathrm{d}\mathcal{X} \; + \; f(-a_1 i_k) \int\limits_{\mathfrak{O}_\nu} \int\limits_{\pi^{k-(X_2-X_1)}\mathfrak{O}_\nu \setminus \pi^{k+1}\mathfrak{O}_\nu} f(\xi x_1) \, \mathrm{d}\mathcal{X}$$
$$+ f(-a_1 i_k) f(\xi i_k) \int_{\mathfrak{O}_\nu} \int_{\pi^{k+1}\mathfrak{O}_\nu} \mathrm{d}\mathcal{X}$$

Disregarding the first term by Lemma 5.2.2, this becomes

$$= \frac{1}{m}(\rho^{(X_2-X_1)+1} - 1) f(-a_1 i_k) + f(-a_1 i_k) f(\xi i_k).$$

Now, using Lemma 5.2.3, when summing over the $i_k$'s we find,

$$\sum_{i_k \in \Pi \setminus 0} I_1^{(\xi)}(i_k) = \frac{(\rho - 1)}{m}(X_2 - X_1 + 1) + \sum_{i_k \in \Pi \setminus 0} f(-a_1 i_k) f(\xi i_k).$$

**Case (iii)** : $X_2 - X_1 < -1 < k$ :

$$I_1^{(\xi)}(i_k) = \int\limits_{\mathfrak{O}_\nu} \int\limits_{\mathfrak{O}_\nu \setminus \pi^{k+1}\mathfrak{O}_\nu} f(a_2 x_1) f(\xi x_1) \mathrm{d}\mathcal{X} \; + \; f(a_2 i_k) \int\limits_{\mathfrak{O}_\nu} \int\limits_{\pi^{k+1}\mathfrak{O}_\nu \setminus \pi^{k-(X_2-X_1)}\mathfrak{O}_\nu} f(\xi x_1) \, \mathrm{d}\mathcal{X}$$
$$+ f(-a_1 i_k) f(\xi i_k) \int_{\mathfrak{O}_\nu} \int_{\pi^{k-(X_2-X_1)}\mathfrak{O}_\nu} \mathrm{d}\mathcal{X}$$

Once again, by Lemma 5.2.2, we may write

$$= \frac{1}{m}(\rho^{(X_2-X_1)+1} - 1) f(a_2 i_k) + f(-a_1 i_k) f(\xi i_k).$$

Finally, summing over the $i_k$'s, in this case we find

$$\sum_{i_k \in \Pi \setminus 0} I_1^{(\xi)}(i_k) = \frac{(\rho-1)}{m}(X_2 - X_1 + 1) + \sum_{i_k \in \Pi \setminus 0} f(-a_1 i_k) f(\xi i_k).$$

Since Case (ii) and Case (iii) are identical the "$-1$" mentioned earlier makes it convenient for us to combine them into a single case when evaluating the cocycle.

**Case (ii,iii)** : $X_2 - X_1 < k$ :

$$\sum_{i_k \in \Pi \setminus 0} I_1^{(\xi)}(i_k) = \frac{(\rho-1)}{m}(X_2 - X_1 + 1) + \sum_{i_k \in \Pi \setminus 0} f(-a_1 i_k) f(\xi i_k).$$

### The Integrals $I_2^{(\xi)}(i_k)$

As with the proof of the previous theorem we shall employ the substitutions,

$$\begin{aligned} x_1 &\longmapsto x_1 + \pi^{-Z_2} c_2(\pi^{Z_2} x_2 + \pi^{Z_2 - 1} i_{Z_2 - 1} + \ldots + \pi^k i_k) \\ x_2 &\longmapsto \pi^{Z_2} x_2 + \pi^{Z_2 - 1} i_{Z_2 - 1} + \ldots + \pi^k i_k \\ \Rightarrow \quad \mathrm{d}\mathcal{X} &\longmapsto \rho^{-Z_2} \mathrm{d}\mathcal{X} \equiv \mathrm{d}\mathcal{X} \pmod{m}. \end{aligned}$$

These substitutions allow us to calculate,

$$\begin{aligned} I_2^{(\xi)}(i_k) &= \int_{\mathfrak{O}_\nu^2} f\Big(\begin{smallmatrix} -\pi^{X_1} a_1(\pi^{Z_2} x_2 + \ldots + \pi^k i_k) \\ \pi^{X_2} a_2(x_1 + \pi^{-Z_2} c_2(\pi^{Z_2} x_2 + \ldots + \pi^k i_k)) \end{smallmatrix}\Big) f\Big(\begin{smallmatrix} \xi(x_1 + \pi^{-Z_2} c(\pi^{Z_2} x_2 + \ldots + \pi^k i_k)) \\ \xi(\pi^{Z_2} x_2 + \ldots + \pi^k i_k) \end{smallmatrix}\Big) \mathrm{d}\mathcal{X} \\ &= f(\xi c_2 i_k) \int_{\mathfrak{O}_\nu^2} f\Big(\begin{smallmatrix} -\pi^{X_1 + k} a_1 i_k \\ \pi^{X_2 - (Z_2 - k)} a_2 c_2 i_k \end{smallmatrix}\Big) \mathrm{d}\mathcal{X}. \end{aligned}$$

This will give us two distinct cases to consider.

**Case (1)** : $X_2 - X_1 \geq Z_2$

Since $X_1 + k \leq X_2 - (Z_2 - k)$ we find,

$$\sum_{i_k \in \Pi \setminus 0} I_2^{(\xi)}(i_k) = \sum_{i_k \in \Pi \setminus 0} f(-a_1 i_k) f(\xi c_2 i_k).$$

**Case (2)** $:$ $X_2 - X_1 < Z_2$ :

In this case we have $X_1 + k > X_2 - (Z_2 - k)$. Therefore we have,

$$\sum_{i_k \in \Pi \backslash 0} I_2^{(\xi)}(i_k) = \sum_{i_k \in \Pi \backslash 0} f(a_2 c_2 i_k) f(\xi c_2 i_k).$$

Referring to our work on the integrals $I_1^{(\xi)}(i_k)$ and $I_2^{(\xi)}(i_k)$ we see that, when calculating the cocycle $dec_\nu$, we shall have three distinct cases to consider. Using the results given in these sections together with Lemmas 5.2.2, 5.2.3 and 2.0.5 we shall once again go through each of these cases in detail.

**Results for** $X_2 - X_1 < 0 < Z_2$ :

| | $0 \le k < Z_2$ |
|---|---|
| $I_1^{(\xi)}(i_k)$ | Case (ii,iii) |
| $I_2^{(\xi)}(i_k)$ | Case (2) |

Using the results given in Case (ii,iii) and Case (2) we find,

$$dec_\nu(\alpha_2 \eta_\psi, n_2) = \prod_{k=0}^{Z_2-1} \prod_{\xi \in \mu_m} \xi^{\sum_{i_k} I_1^{(\xi)}(i_k)} \prod_{k=0}^{Z_2-1} \prod_{\xi \in \mu_m} \xi^{-\sum_{i_k} I_2^{(\xi)}(i_k)}$$

$$= \prod_{k=0}^{Z_2-1} \Big( \prod_{\xi \in \mu_m} \xi \Big)^{\frac{(p-1)}{m}(X_2 - X_1 + 1)} \prod_{k=0}^{Z_2-1} \prod_{\xi \in \mu_m} \xi^{\sum_{i_k} f(-a_1 i_k) f(\xi i_k)} \prod_{k=0}^{Z_2-1} \prod_{\xi \in \mu_m} \xi^{-\sum_{i_k} f(a_2 c_2 i_k) f(\xi c_2 i_k)}$$

$$= (-1)^{\frac{(p-1)}{m} Z_2 (X_2 - X_1 + 1)} (-a_1, \pi)_{\nu,m}^{Z_2} (a_2, \pi)_{\nu,m}^{-Z_2}$$

$$= (a_1/a_2, \pi)_{\nu,m}^{Z_2} (-1)^{\frac{(p-1)}{2^r} Z_2 (X_2 - X_1)}.$$

**Results for** $0 \le X_2 - X_1 < Z_2$ :

| | $0 \le k < X_2 - X_1$ | $X_2 - X_1 \le k < Z_2$ |
|---|---|---|
| $I_1^{(\xi)}(i_k)$ | Case (i) | Case (ii,iii) |
| $I_2^{(\xi)}(i_k)$ | Case (2) | |

For this case we shall again need to split the product over $k$ of the integrals $I_1^{(\xi)}(i_k)$ into the two possible cases depending on the value of $k$. Using our previous results we find,

$$dec_\nu(\alpha_2\eta_\psi, n_2) = \prod_{k=0}^{X_2-X_1}\prod_{\xi\in\mu_m}\xi^{\sum_{i_k} I_1^{(\xi)}(i_k)} \prod_{k=X_2-X_1+1}^{Z_2-1}\prod_{\xi\in\mu_m}\xi^{\sum_{i_k} I_1^{(\xi)}(i_k)} \prod_{k=0}^{Z_2-1}\prod_{\xi\in\mu_m}\xi^{-\sum_{i_k} I_2^{(\xi)}(i_k)}$$

$$= \Big(\prod_{\xi\in\mu_m}\xi\Big)^{\frac{(\rho-1)}{m}\sum_{k=0}^{X_2-X_1}(k+1)} \prod_{k=0}^{X_2-X_1}\prod_{\xi\in\mu_m}\xi^{\sum_{i_k} f(-a_1 i_k)f(\xi i_k)}$$

$$\prod_{k=X_2-X_1}^{Z_2-1}\Big(\prod_{\xi\in\mu_m}\xi\Big)^{\frac{(\rho-1)}{m}(X_2-X_1+1)} \prod_{k=X_2-X_1}^{Z_2-1}\prod_{\xi\in\mu_m}\xi^{\sum_{i_k} f(-a_1 i_k)f(\xi i_k)}$$

$$\prod_{k=0}^{Z_2-1}\prod_{\xi\in\mu_m}\xi^{-\sum_{i_k} f(a_2 c_2 i_k)f(\xi c_2 i_k)}$$

$$= (-1)^{\frac{(\rho-1)}{m}\frac{(X_2-X_1+1)(X_2-X_1+2)}{2}}(-a_1,\pi)_{\nu,m}^{X_2-X_1+1}$$

$$(-1)^{\frac{(\rho-1)}{m}(X_2-X_1+1)(Z_2-(X_2-X_1+1))}(-a_1,\pi)_{\nu,m}^{Z_2-(X_2-X_1+1)}(a_2,\pi)_{\nu,m}^{-Z_2}$$

$$= (a_1/a_2,\pi)_{\nu,m}^{Z_2}\,(-1)^{\frac{(\rho-1)}{2^r}Z_2(X_2-X_1)}(-1)^{\frac{(\rho-1)}{2^r}\frac{(X_2-X_1)(X_2-X_1+1)}{2}}.$$

**Results for** $0 < Z_2 \le X_2 - X_1$ **:**

| | $0 \le k < Z_2$ |
|---|---|
| $I_1^{(\xi)}(i_k)$ | Case (i) |
| $I_2^{(\xi)}(i_k)$ | Case (1) |

Using the results of Case (i) and Case (1) we find,

$$dec_\nu(\alpha_2\eta_\psi, n_2) = \prod_{k=0}^{Z_2-1}\prod_{\xi\in\mu_m}\xi^{\sum_{i_k} I_1^{(\xi)}(i_k)} \prod_{k=0}^{Z_2-1}\prod_{\xi\in\mu_m}\xi^{-\sum_{i_k} I_2^{(\xi)}(i_k)}$$

$$= \Big(\prod_{\xi\in\mu_m}\xi\Big)^{\frac{(\rho-1)}{m}\sum_{k=0}^{Z_2-1}(k+1)} \prod_{k=0}^{Z_2-1}\prod_{\xi\in\mu_m}\xi^{\sum_{i_k} f(-a_1 i_k)f(\xi i_k)} \prod_{k=0}^{Z_2-1}\prod_{\xi\in\mu_m}\xi^{-\sum_{i_k} f(-a_1 i_k)f(\xi c_2 i_k)}$$

$$= (-1)^{\frac{(\rho-1)}{m}\frac{Z_2(Z_2+1)}{2}}.(-a_1,\pi)_{\nu,m}^{Z_2}.(-a_1/c_2,\pi)_{\nu,m}^{-Z_2}$$

$$= (c_2,\pi)_{\nu,m}^{Z_2}\,(-1)^{\frac{(\rho-1)}{2^r}\frac{Z_2(Z_2+1)}{2}},$$

which completes our proof.

□

**Corollary 3** *For each* $n_2 := n_{1,2}(\pi^{-Z_2}c_2) \in N$ *and* $\eta_\psi \in \mathfrak{M}$ *where* $\psi = s_{(1,2)} \in W$ *the cocycle* $dec_\nu$ *satisfies,*

$$dec_\nu\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & \pi^{-Z_2}c_2 \\ 0 & 1 \end{smallmatrix}\right)\right) = 1.$$

## 5.5 $dec_\nu$ on $N.T \times N$

**Theorem 5.5.1** *Let* $\alpha_2 \in T$ *and* $n_2 := n_{1,2}(\pi^{-Z_2}c_2) \in N$ *be as previously defined. Let us also define* $n_1 := n_{1,2}(\pi^{-Z_1}c_1) \in N$. *Then, whenever* $Z_1 < 0$ *the cocycle* $dec_\nu$ *satisfies,*

$$dec_\nu\left(\left(\begin{smallmatrix} 1 & \pi^{-Z_1}c_1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} \pi^{X_1}a_1 & 0 \\ 0 & \pi^{X_2}a_2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & \pi^{-Z_2}c_2 \\ 0 & 1 \end{smallmatrix}\right)\right) = dec_\nu\left(\left(\begin{smallmatrix} \pi^{X_1}a_1 & 0 \\ 0 & \pi^{X_2}a_2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & \pi^{-Z_2}c_2 \\ 0 & 1 \end{smallmatrix}\right)\right).$$

**PROOF OF THEOREM:**

To prove this result we must simply notice that whenever $Z_1 < 0$ the matrix $n_1$ satisfies,

$$n_1 := \left(\begin{smallmatrix} 1 & \pi^{-Z_1}c_1 \\ 0 & 1 \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) = I_2 \pmod{\pi}.$$

Therefore, by Lemma 1.4.4, we have $fn_1 = f$ and it immediately follows that,

$$\begin{aligned} dec_\nu(n_1\alpha_2, n_2) &= < fn_1\alpha_2 - f|M - Mn_2^{-1} > \\ &= < f\alpha_2 - f|M - Mn_2^{-1} > \\ &= dec_\nu(\alpha_2, n_2). \end{aligned}$$

□

**Theorem 5.5.2** *Let* $\alpha_2 \in T$ *and also let,*

$$n_1 := n_{1,2}(\pi^{-Z_1}c_1), \quad n_2 := n_{1,2}(\pi^{-Z_2}c_2) \in N,$$

*with* $Z_1 \geq 0$, $Z_2 > 0$. *Then having defined,*

$$(a_1c_2 + c_1a_2) := \pi^D d,$$

*for some* $D \geq 0$ *and where* $d = 0$ *or* $|\,d\,|_\nu = 1$, *the cocycle* $dec_\nu$ *satisfies,*

• $Z_2 < (X_1 - X_2) + Z_1$ :

$$dec_\nu(n_1\alpha_2, n_2) = (c_2, \pi)^{Z_2}_{\nu,m} (-1)^{\frac{(\rho-1)}{2^r}\frac{Z_2(Z_2+1)}{2}}$$

• $(X_1 - X_2) + Z_1 < 0$ :

$$dec_\nu(n_1\alpha_2, n_2) = (a_2/a_1, \pi)^{Z_2}_{\nu,m} \quad (-1)^{\frac{(\rho-1)}{2^r}Z_2(X_1-X_2)}$$

• $0 \leq (X_1 - X_2) + Z_1 < Z_2$ :

$$\begin{aligned} dec_\nu(n_1\alpha_2, n_2) = {} & (a_2/a_1, \pi)^{Z_2}_{\nu,m} (-1)^{\frac{(\rho-1)}{2^r}Z_2(X_1-X_2)} \\ & (c_1, \pi)^{(X_1-X_2)+Z_1}_{\nu,m} (-1)^{\frac{(\rho-1)}{2^r}\frac{(X_1-X_2)((X_1-X_2)-1)}{2}} (-1)^{\frac{(\rho-1)}{2^r}\frac{Z_1(Z_1+1)}{2}} \end{aligned}$$

• $X_1 - X_2 = Z_2 - Z_1$ :

∘ $(a_1c_2 + c_1a_2) = 0$ *or* $D \not\leq Z_1, Z_2$ :

$Z_2 \leq Z_1$ : $$dec_\nu(n_1\alpha_2, n_2) = (c_1c_2, \pi)^{Z_2}_{\nu,m}(-1)^{\frac{(\rho-1)}{2^r}Z_2}(-1)^{\frac{(\rho-1)}{2^r}Z_1Z_2}$$

$Z_1 \leq Z_2$ : $$dec_\nu(n_1\alpha_2, n_2) = (c_1c_2, \pi)^{Z_2}_{\nu,m}(-1)^{\frac{(\rho-1)}{2^r}\frac{Z_2(Z_2+1)}{2}}(-1)^{\frac{(\rho-1)}{2^r}\frac{Z_1(Z_1+1)}{2}}$$

∘ $(a_1c_2 + c_1a_2) \neq 0$, $D \leq Z_1, Z_2$ :

$$\begin{aligned} dec_\nu(n_1\alpha_2, n_2) = {} & (c_1c_2, \pi)^{Z_2}_{\nu,m}(a_2/d, \pi)^{Z_2-D}_{\nu,m} \\ & (-1)^{\frac{(\rho-1)}{2^r}\frac{Z_2(Z_2+1)}{2}}(-1)^{\frac{(\rho-1)}{2^r}\frac{D(D-1)}{2}}(-1)^{\frac{(\rho-1)}{2^r}Z_1D}. \end{aligned}$$

Before we continue with the proof of this theorem we shall first consider some calculations which shall appear repeatedly throughout the course of this proof and the next. We shall present these calculations as three lemmas and then simply refer to them as required. However, before we can continue we shall first require the following:

**Remark:**

At this point we note that we may discontinuously extend the function $f$, given in section 1.4.3, to the whole of $k_\nu$.

For each $\pi^X a \in k_\nu \backslash 0$ with $|a|_\nu = 1$ we had previously defined the function $f : k_\nu \backslash 0 \to \mathbb{Z}$ by,

$$f(\pi^X a) = f_{\mathcal{S}}(a \pmod{\pi}) = \begin{cases} 1 & a \in \mathcal{S} \\ 0 & a \notin \mathcal{S}, \end{cases}$$

where $\mathcal{S}$ is a complete set of representatives for $\mu_m$-orbits in $(\mathfrak{O}_\nu/\wp_\nu) \setminus 0$.

Now, let us define the discontinuous function $f_0 : (\mathfrak{O}_\nu/\wp_\nu) \longrightarrow \mathbb{Z}$ by,

$$f_0(a) = \begin{cases} f_{\mathcal{S}}(a) & a \neq 0 \\ 0 & a = 0. \end{cases}$$

Finally, we may discontinuously extend the function $f$ to the whole of $k_\nu$, by re-defining

$$f(\pi^X a) = f_0(a \pmod{\pi}), \quad \text{for each } a \in \mathfrak{O}_\nu/\pi\mathfrak{O}_\nu.$$

For the following lemmas, as with the integrals $I_1^{(\xi)}(i_k)$ and $I_2^{(\xi)}(i_k)$, we shall once again abuse the notation with respect to the equals sign. That is, we shall use "=" to indicate that two expressions are equal when considered to be exponents of the product of roots.

**Lemma 5.5.1** *For any non-zero $\lambda_1,\ \lambda_2 \in \mathfrak{O}_\nu/\pi\mathfrak{O}_\nu$ we define the integral $K_1^{(\xi)}(i_k)$ by,*

$$K_1^{(\xi)}(i_k) = \int_{\substack{x_1 \in \mathfrak{O}_\nu^\times \\ \lambda_1 x_1 + \lambda_2 i_k \not\equiv 0(\pi)}} f(\lambda_1 x_1 + \lambda_2 i_k) f(\xi x_1) \mathrm{d}x_1.$$

*Then this integral $K_1^{(\xi)}(i_k)$ satisfies,*

$$\sum_{i_k \in \Pi\backslash 0} K_1^{(\xi)}(i_k) = \frac{(\rho - 1)}{m} - \sum_{x_1 \in \Pi\backslash 0} f(\lambda_1 x_1) f(\xi x_1).$$

**PROOF:**

Referring to the previous remark concerning the extension of our function $f$ we are able to write,

$$\begin{aligned} K_1^{(\xi)}(i_k) &= \int_{\substack{x_1 \in \mathfrak{O}_\nu^\times \\ \lambda_1 x_1 + \lambda_2 i_k \not\equiv 0(\pi)}} f(\lambda_1 x_1 + \lambda_2 i_k) f(\xi x_1) \mathrm{d}x_1 \\ &= \sum_{\substack{x_1 \in \Pi \\ x_1 \neq 0, -\lambda_1^{-1}\lambda_2 i_k}} f(\lambda_1 x_1 + \lambda_2 i_k) f(\xi x_1) \\ &= \sum_{x_1 \in \Pi\backslash 0} f_0(\lambda_1 x_1 + \lambda_2 i_k) f_0(\xi x_1). \end{aligned}$$

Thus, when we sum over the $i_k$, we find that

$$\begin{aligned}
\sum_{i_k \in \Pi\backslash 0} K_1^{(\xi)}(i_k) &= \sum_{x_1 \in \Pi\backslash 0} f_0(\xi x_1) \sum_{i_k \in \Pi\backslash 0} f_0(\lambda_1 x_1 + \lambda_2 i_k) \\
&= \sum_{x_1 \in \Pi\backslash 0} f_0(\xi x_1)\Big(\sum_{x \in \Pi} f_0(x) - f_0(\lambda_1 x_1)\Big) \\
&= \sum_{x_1 \in \Pi\backslash 0} f_0(\xi x_1) \sum_{x \in \Pi} f_0(x) - \sum_{x_1 \in \Pi\backslash 0} f_0(\lambda_1 x_1) f_0(\xi x_1) \\
&= \frac{(\rho-1)}{m} - \sum_{x_1 \in \Pi\backslash 0} f(\lambda_1 x_1) f(\xi x_1). \qquad \square
\end{aligned}$$

**Lemma 5.5.2** *For any non-zero $\lambda_1$, $\lambda_2$, $\lambda_3 \in \mathfrak{O}_\nu/\pi\mathfrak{O}_\nu$ we define the integral $K_2^{(\xi)}(i_k)$ by,*

$$K_2^{(\xi)}(i_k) = \int_{\substack{x_1 \in \mathfrak{O}_\nu^\times \\ \lambda_1 x_1 + \lambda_2 i_k \not\equiv 0 (\pi)}} f(\lambda_1 x_1 + \lambda_2 i_k) f(\xi \lambda_3 i_k) \mathrm{d}x_1.$$

*Then this integral $K_2^{(\xi)}(i_k)$ satisfies,*

$$\sum_{i_k \in \Pi\backslash 0} K_2^{(\xi)}(i_k) = \frac{(\rho-1)}{m} - \sum_{i_k \in \Pi\backslash 0} f(\lambda_2 i_k) f(\xi \lambda_3 i_k).$$

**PROOF:**

As with the previous lemma, using the extension of our function $f$, we have

$$\begin{aligned}
K_2^{(\xi)}(i_k) &= \int_{\substack{x_1 \in \mathfrak{O}_\nu^\times \\ \lambda_1 x_1 + \lambda_2 i_k \not\equiv 0 (\pi)}} f(\lambda_1 x_1 + \lambda_2 i_k) f(\xi \lambda_3 i_k) \mathrm{d}x_1 \\
&= f(\xi \lambda_3 i_k) \sum_{\substack{x_1 \in \Pi \\ x_1 \not\equiv 0, -\lambda_1^{-1} \lambda_2 i_k}} f(\lambda_1 x_1 + \lambda_2 i_k) \\
&= f_0(\xi \lambda_3 i_k) \sum_{x_1 \in \Pi\backslash 0} f_0(\lambda_1 x_1 + \lambda_2 i_k).
\end{aligned}$$

Therefore, when we sum over the $i_k$, we now find

$$\begin{aligned}
\sum_{i_k \in \Pi\backslash 0} K_2^{(\xi)}(i_k) &= \sum_{i_k \in \Pi\backslash 0} f(\xi \lambda_3 i_k) \sum_{x_1 \in \Pi\backslash 0} f_0(\lambda_1 x_1 + \lambda_2 i_k) \\
&= \sum_{i_k \in \Pi\backslash 0} f(\xi \lambda_3 i_k) \Big( \sum_{x \in \Pi} f_0(x) - f_0(\lambda_2 i_k) \Big) \\
&= \frac{(\rho-1)}{m} - \sum_{i_k \in \Pi\backslash 0} f(\lambda_2 i_k) f(\xi \lambda_3 i_k). \qquad \square
\end{aligned}$$

**Lemma 5.5.3** *For any non-zero $\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4 \in \mathfrak{O}_\nu/\pi\mathfrak{O}_\nu$ and any $\Lambda \in \mathbb{Z}^{\geq 0}$ we define the integral $K_3^{(\xi)}(i_k)$ by,*

$$K_3^{(\xi)}(i_k) = \int_{\mathfrak{O}_\nu} \int_{\substack{x_1 \in \mathfrak{O}_\nu^\times \\ \lambda_1 x_1 + \lambda_2 i_k \equiv 0(\pi)}} f\left(\tfrac{(\lambda_1 x_1 + \lambda_2 i_k) + \pi\lambda_2 i_{k+1} + \ldots + \pi^{Z_2 - k}\lambda_2 x_2}{\pi^\Lambda \lambda_4 i_k}\right) f(\xi\lambda_3 i_k) \mathrm{d}\mathcal{X}.$$

*Then this integral satisfies,*

$$\sum_{i_k \in \Pi\backslash 0} K_3^{(\xi)}(i_k) = \frac{(\rho - 1)}{m}\Lambda + \sum_{i_k \in \Pi\backslash 0} f(\lambda_4 i_k) f(\xi\lambda_3 i_k).$$

**PROOF:**

Having made the substitution,

$$x_1 \longmapsto \pi x_1 - \lambda_1^{-1}\lambda_2 i_k, \qquad \mathrm{d}x_1 \longmapsto \mathrm{d}x_1 \pmod m,$$

we have,

$$\begin{aligned} K_3^{(\xi)}(i_k) &= \int_{\mathfrak{O}_\nu} \int_{\mathfrak{O}_\nu} f\left(\tfrac{\pi(\lambda_1 x_1 + \lambda_2 i_{k+1}) + \pi^2\lambda_2 i_{k+2} + \ldots + \pi^{Z_2 - k}\lambda_2 x_2}{\pi^\Lambda \lambda_4 i_k}\right) f(\xi\lambda_3 i_k) \mathrm{d}\mathcal{X} \\ &= \int_{\mathfrak{O}_\nu} \int_{\mathfrak{O}_\nu} f\left(\tfrac{\pi(\lambda_1 x_1 + \lambda_2 i_{k+1} + \pi\lambda_2 i_{k+2} + \ldots + \pi^{Z_2 - k - 1}\lambda_2 x_2)}{\pi^\Lambda \lambda_4 i_k}\right) f(\xi\lambda_3 i_k) \mathrm{d}\mathcal{X}. \end{aligned}$$

which with a simple change of variables,

$$x_1 \longmapsto x_1, \qquad (\lambda_1 x_1 + \lambda_2 i_{k+1} + \pi\lambda_2 i_{k_2} + \ldots + \pi^{Z_2 - k - 1}\lambda_2 x_2) \longmapsto x$$
$$\mathrm{d}\mathcal{X} \longmapsto \mathrm{d}x_1 \mathrm{d}x \pmod m,$$

becomes,

$$\begin{aligned} K_3^{(\xi)}(i_k) &= \int_{\mathfrak{O}_\nu} \int_{\mathfrak{O}_\nu} f\left(\tfrac{\pi x}{\pi^\Lambda \lambda_4 i_k}\right) f(\xi\lambda_3 i_k) \mathrm{d}x_1 \mathrm{d}x \\ &= f(\xi\lambda_3 i_k) \int_{\mathfrak{O}_\nu \backslash \pi^\Lambda \mathfrak{O}_\nu} f(x) \mathrm{d}x + f(\lambda_4 i_k) f(\xi\lambda_3 i_k) \int_{\pi^\Lambda \mathfrak{O}_\nu} \mathrm{d}x \\ &= f(\xi\lambda_3 i_k) \frac{1}{m}(\rho^\Lambda - 1) + f(\lambda_4 i_k) f(\xi\lambda_3 i_k). \end{aligned}$$

In conclusion we find that when we sum over the $i_k$ we have,

$$\sum_{i_k \in \Pi\backslash 0} K_3^{(\xi)}(i_k) = \frac{(\rho - 1)}{m}\Lambda + \sum_{i_k \in \Pi\backslash 0} f(\lambda_4 i_k) f(\xi\lambda_3 i_k).$$

□

**Proof of Theorem 5.5.2:**

Once again we shall begin by considering the integrals $I_1^{(\xi)}(i_k)$ and $I_2^{(\xi)}(i_k)$.

### The Integrals $I_1^{(\xi)}(i_k)$

Having applied the usual substitution for the integrals $I_1^{(\xi)}(i_k)$ we see that these are indeed only dependent on $i_k$ and are given by,

$$I_1^{(\xi)}(i_k) = \int_{\mathfrak{O}_\nu^2} f\genfrac{(}{)}{0pt}{}{\pi^{X_1}a_1x_1+\pi^{-Z_1+X_2}c_1a_2(\pi^{Z_2}x_2+\ldots+\pi^k i_k)}{\pi^{X_2}a_2(\pi^{Z_2}x_2+\ldots+\pi^k i_k)} f\genfrac{(}{)}{0pt}{}{\xi x_1}{\xi\pi^k i_k}\mathrm{d}\mathcal{X}.$$

After careful consideration we see that in order to calculate this integral we shall need to consider three distinct cases.

(i) $0 \le k < (X_1 - X_2) + Z_1$, (ii) $0 \le (X_1 - X_2) + Z_1 \le k$, (iii) $(X_1 - X_2) + Z_1 < 0 \le k$.

### Case (i) : $0 \le k < (X_1 - X_2) + Z_1$

In this case we shall have $-Z_1 + X_2 + k < X_1$ and $-Z_1 + X_2 + k \le X_2 + k$. Therefore we simply find that,

$$\begin{aligned}
I_1^{(\xi)}(i_k) &= \int_{\mathfrak{O}_\nu^2} f\genfrac{(}{)}{0pt}{}{\pi^{X_1}a_1x_1+\pi^{-Z_1+X_2}c_1a_2(\pi^{Z_2}x_2+\ldots+\pi^k i_k)}{\pi^{X_2}a_2(\pi^{Z_2}x_2+\ldots+\pi^k i_k)} f\genfrac{(}{)}{0pt}{}{\xi x_1}{\xi\pi^k i_k}\mathrm{d}\mathcal{X} \\
&= f(c_1a_2i_k)\int_{\mathfrak{O}_\nu^2} f\genfrac{(}{)}{0pt}{}{\xi x_1}{\xi\pi^k i_k}\mathrm{d}\mathcal{X} \\
&= f(c_1a_2i_k)\int_{\mathfrak{O}_\nu\setminus\pi^{k+1}\mathfrak{O}_\nu} f(\xi x_1)\mathrm{d}x_1 \; + \; f(c_1a_2i_k)f(\xi i_k)\int_{\pi^{k+1}\mathfrak{O}_\nu}\mathrm{d}x_1 \\
&= \frac{1}{m}(\rho^{k+1}-1)f(c_1a_2i_k) + f(c_1a_2i_k)f(\xi i_k).
\end{aligned}$$

Finally, taking the sum over the $i_k$, we find

$$\sum_{i_k\in\Pi\setminus 0} I_1^{(\xi)}(i_k) = \frac{(\rho-1)}{m}(k+1) + \sum_{i_k\in\Pi\setminus 0} f(c_1a_2i_k)f(\xi i_k).$$

**Case (ii) : $0 \le (X_1 - X_2) + Z_1 \le k$**

In this more complicated case we find that our integral satisfies,

$$\begin{aligned}
I_1^{(\xi)}(i_k) &= \int_{\mathfrak{O}_\nu \setminus \pi^{k-Z_1-(X_1-X_2)}\mathfrak{O}_\nu} f(a_1x_1)f(\xi x_1)\mathrm{d}x_1 \\
&\quad + \int\limits_{\mathfrak{O}_\nu} \int\limits_{\pi^{k-Z_1-(X_1-X_2)}\mathfrak{O}_\nu \setminus \pi^{k-Z_1-(X_1-X_2)+1}\mathfrak{O}_\nu} f\left(\tfrac{\pi^{X_1}a_1x_1+\pi^{-Z_1+X_2}c_1a_2(\pi^{Z_2}x_2+\ldots+\pi^k i_k)}{\pi^{X_2}a_2(\pi^{Z_2}x_2+\ldots+\pi^k i_k)}\right) f(\xi x_1)\mathrm{d}\mathcal{X} \\
&\quad + f(c_1a_2i_k)\int_{\pi^{k-Z_1-(X_1-X_2)+1}\mathfrak{O}_\nu \setminus \pi^{k+1}\mathfrak{O}_\nu} f(\xi x_1)\mathrm{d}x_1 \\
&\quad + f(c_1a_2i_k)f(\xi i_k)\int_{\pi^{k+1}\mathfrak{O}_\nu} \mathrm{d}x_1.
\end{aligned}$$

The first term in this expression may be disregarded by Lemma 5.2.2 while the third and fourth terms are easy to compute. Therefore, in order to calculate the integral $I_1^{(\xi)}(i_k)$ in this case we need to find

$$\tilde{I_1}^{(\xi)}(i_k) := \int\limits_{\mathfrak{O}_\nu} \int\limits_{\pi^{k-Z_1-(X_1-X_2)}\mathfrak{O}_\nu \setminus \pi^{k-Z_1-(X_1-X_2)+1}\mathfrak{O}_\nu} f\left(\tfrac{\pi^{X_1}a_1x_1+\pi^{-Z_1+X_2}c_1a_2(\pi^{Z_2}x_2+\ldots+\pi^k i_k)}{\pi^{X_2}a_2(\pi^{Z_2}x_2+\ldots+\pi^k i_k)}\right) f(\xi x_1)\mathrm{d}\mathcal{X}.$$

Having made the obvious substitution,

$$x_1 \longmapsto \pi^{k-Z_1-(X_1-X_2)}x_1, \qquad \mathrm{d}x_1 \longmapsto \mathrm{d}x_1 \pmod{m},$$

this becomes,

$$\begin{aligned}
\tilde{I_1}^{(\xi)}(i_k) &= \int_{\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu^\times} f\left(\tfrac{\pi^{k-Z_1+X_2}(a_1x_1+c_1a_2i_k+\pi c_1a_2i_{k+1}+\ldots+\pi^{Z_2-k}c_1a_2x_2)}{\pi^{X_2+k}a_2i_k}\right) f(\xi x_1)\mathrm{d}\mathcal{X} \\
&= \int_{\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu^\times} f\left(\tfrac{(a_1x_1+c_1a_2i_k)+\pi c_1a_2i_{k+1}+\ldots+\pi^{Z_2-k}c_1a_2x_2}{\pi^{Z_1}a_2i_k}\right) f(\xi x_1)\mathrm{d}\mathcal{X} \\
&= \int\limits_{\substack{\mathfrak{O}_\nu \\ a_1x_1+c_1a_2i_k\not\equiv 0(\pi)}}\int_{x_1\in\mathfrak{O}_\nu^\times} f(a_1x_1+c_1a_2i_k)f(\xi x_1)\mathrm{d}\mathcal{X} \\
&\quad + \int\limits_{\substack{\mathfrak{O}_\nu \\ a_1x_1+c_1a_2i_k\equiv 0(\pi)}}\int_{x_1\in\mathfrak{O}_\nu^\times} f\left(\tfrac{(a_1x_1+c_1a_2i_k)+\pi c_1a_2i_{k+1}+\ldots+\pi^{Z_2-k}c_1a_2x_2}{\pi^{Z_1}a_2i_k}\right) f(\xi x_1)\mathrm{d}\mathcal{X}.
\end{aligned}$$

However, in the last term of this expression we have $a_1x_1 + c_1a_2i_k \equiv 0\,(\pi)$, from which we may deduce,

$$f(\xi x_1) = f(\xi(-a_1^{-1}c_1a_2i_k)).$$

This enables us to write,

$$\tilde{I}_1^{(\xi)}(i_k) = K_1^{(\xi)}(i_k) + K_3^{(\xi)}(i_k),$$

where the integrals $K_1^{(\xi)}(i_k)$ and $K_3^{(\xi)}(i_k)$ are those described in Lemmas 5.5.1 and 5.5.3 respectively.

Using the results proved in those lemmas and changing the necessary constants we are immediately able to conclude that,

$$\begin{aligned}
\sum_{i_k \in \Pi \backslash 0} \tilde{I}_1^{(\xi)}(i_k) &= \sum_{i_k \in \Pi \backslash 0} K_1^{(\xi)}(i_k) + \sum_{i_k \in \Pi \backslash 0} K_3^{(\xi)}(i_k) \\
&= \left( \frac{(\rho-1)}{m} - \sum_{x_1 \in \Pi \backslash 0} f(a_1 x_1) f(\xi x_1) \right) + \left( \frac{(\rho-1)}{m} Z_1 + \sum_{i_k \in \Pi \backslash 0} f(a_2 i_k) f(\xi(-a_1^{-1} c_1 a_2) i_k) \right).
\end{aligned}$$

Finally, having solved for $\tilde{I}_1^{(\xi)}(i_k)$, we are now able to calculate integrals $I_1^{(\xi)}(i_k)$.

That is, whenever $0 \leq (X_1 - X_2) + Z_1 \leq k$, the integrals $I_1^{(\xi)}(i_k)$ satisfy

$$\begin{aligned}
\sum_{i_k \in \Pi \backslash 0} I_1^{(\xi)}(i_k) &= \left( \frac{(\rho-1)}{m}(Z_1+1) - \sum_{x_1 \in \Pi \backslash 0} f(a_1 x_1) f(\xi x_1) + \sum_{i_k \in \Pi \backslash 0} f(a_2 i_k) f(\xi(-a_1^{-1} c_1 a_2) i_k) \right) \\
&\quad + \frac{(\rho-1)}{m}((X_1 - X_2) + Z_1) + \sum_{i_k \in \Pi \backslash 0} f(c_1 a_2 i_k) f(\xi i_k) \\
&= \frac{(\rho-1)}{m}((X_1 - X_2) + 1) - \sum_{x_1 \in \Pi \backslash 0} f(a_1 x_1) f(\xi x_1) \\
&\quad + \sum_{i_k \in \Pi \backslash 0} f(a_2 i_k) f(\xi(-a_1^{-1} c_1 a_2) i_k) + \sum_{i_k \in \Pi \backslash 0} f(c_1 a_2 i_k) f(\xi i_k).
\end{aligned}$$

**Case (iii)** : $(X_1 - X_2) + Z_1 < 0 \leq k$

Returning to the original problem we now find,

$$\begin{aligned}
I_1^{(\xi)}(i_k) &= \int_{\mathfrak{O}_\nu \backslash \pi^{k+1} \mathfrak{O}_\nu} f(a_1 x_1) f(\xi x_1) \mathrm{d}x_1 \\
&\quad + f(\xi i_k) \int_{\pi^{k+1} \mathfrak{O}_\nu \backslash \pi^{k - Z_1 - (X_1 - X_2)} \mathfrak{O}_\nu} f(a_1 x_1) \mathrm{d}x_1 \\
&\quad + f(\xi i_k) \int_{\pi^{k - Z_1 - (X_1 - X_2)} \mathfrak{O}_\nu \backslash \pi^{k - Z_1 - (X_1 - X_2) + 1} \mathfrak{O}_\nu} f\left( \frac{\pi^{X_1} a_1 x_1 + \pi^{-Z_1 + X_2} c_1 a_2 (\pi^{Z_2} x_2 + \ldots + \pi^k i_k)}{\pi^{X_2} a_2 (\pi^{Z_2} x_2 + \ldots + \pi^k i_k)} \right) \mathrm{d}\mathcal{X} \\
&\quad + f(c_1 a_2 i_k) f(\xi i_k) \int_{\pi^{k - Z_1 - (X_1 - X_2) + 1} \mathfrak{O}_\nu} \mathrm{d}x_1.
\end{aligned}$$

Once again, we may ignore the first term. However it is now the second and fourth terms which are easily calculated. Therefore, in order to find the integral $I_1^{(\xi)}(i_k)$ we need concentrate on calculating the term,

$$\tilde{I}_1^{(\xi)}(i_k) := f(\xi i_k) \int_{\pi^{k-Z_1-(X_1-X_2)}\mathfrak{O}_\nu \setminus \pi^{k-Z_1-(X_1-X_2)+1}\mathfrak{O}_\nu} f\Big(\frac{\pi^{X_1}a_1x_1+\pi^{-Z_1+X_2}c_1a_2(\pi^{Z_2}x_2+\ldots+\pi^k i_k)}{\pi^{X_2}a_2(\pi^{Z_2}x_2+\ldots+\pi^k i_k)}\Big)\,\mathrm{d}\mathcal{X}.$$

If we employ the substitution, $x_1 \longmapsto \pi^{k-Z_1-(X_1-X_2)}x_1$, $\mathrm{d}x_1 \longmapsto \mathrm{d}x_1 \pmod{m}$, this expression simply becomes,

$$\begin{aligned}
\tilde{I}_1^{(\xi)}(i_k) &= f(\xi i_k)\int_{\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu^\times} f\Big(\frac{\pi^{k-Z_1+X_2}(a_1x_1+c_1a_2i_k+\pi c_1a_2i_{k+1}+\ldots+\pi^{Z_2-k}c_1a_2x_2)}{\pi^{X_2+k}a_2i_k}\Big)\mathrm{d}\mathcal{X}\\
&= f(\xi i_k)\int_{\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu^\times} f\Big(\frac{(a_1x_1+c_1a_2i_k)+\pi c_1a_2i_{k+1}+\ldots+\pi^{Z_2-k}c_1a_2x_2}{\pi^{Z_1}a_2i_k}\Big)\mathrm{d}\mathcal{X}\\
&= f(\xi i_k)\int_{\mathfrak{O}_\nu}\int_{\substack{x_1\in\mathfrak{O}_\nu^\times\\ a_1x_1+c_1a_2i_k\not\equiv 0(\pi)}} f(a_1x_1+c_1a_2i_k)\mathrm{d}\mathcal{X}\\
&\quad + f(\xi i_k)\int_{\mathfrak{O}_\nu}\int_{\substack{x_1\in\mathfrak{O}_\nu^\times\\ a_1x_1+c_1a_2i_k\equiv 0(\pi)}} f\Big(\frac{(a_1x_1+c_1a_2i_k)+\pi c_1a_2i_{k+1}+\ldots+\pi^{Z_2-k}c_1a_2x_2}{\pi^{Z_1}a_2i_k}\Big)\mathrm{d}\mathcal{X}\\
&= K_2^{(\xi)}(i_k)+K_3^{(\xi)}(i_k),
\end{aligned}$$

where these are precisely the integrals calculated in Lemmas 5.5.2 and 5.5.3. By changing the necessary constants and using the results proved therein we are able to conclude that,

$$\begin{aligned}
\sum_{i_k\in\Pi\setminus 0}\tilde{I}_1^{(\xi)}(i_k) &= \sum_{i_k\in\Pi\setminus 0}K_2^{(\xi)}(i_k)+\sum_{i_k\in\Pi\setminus 0}K_3^{(\xi)}(i_k)\\
&= \Big(\frac{(\rho-1)}{m}-\sum_{i_k\in\Pi\setminus 0}f(c_1a_2i_k)f(\xi i_k)\Big)+\Big(\frac{(\rho-1)}{m}Z_1+\sum_{i_k\in\Pi\setminus 0}f(a_2i_k)f(\xi i_k)\Big).
\end{aligned}$$

Finally we are able to return to the problem of finding $I_1^{(\xi)}(i_k)$. Using what we have found in this section, whenever $0\le (X_1-X_2)+Z_1\le k$, the integrals $I_1^{(\xi)}(i_k)$ satisfy

$$\begin{aligned}
\sum_{i_k\in\Pi\setminus 0} I_1^{(\xi)}(i_k) &= \frac{(\rho-1)}{m}((X_1-X_2)+Z_1-1)\\
&\quad+\Big(\frac{(\rho-1)}{m}(Z_1+1)-\sum_{i_k\in\Pi\setminus 0}f(c_1a_2i_k)f(\xi i_k)+\sum_{i_k\in\Pi\setminus 0}f(a_2i_k)f(\xi i_k)\Big)\\
&\quad+\sum_{i_k\in\Pi\setminus 0}f(c_1a_2i_k)f(\xi i_k)\\
&= \frac{(\rho-1)}{m}(X_1-X_2)+\sum_{i_k\in\Pi\setminus 0}f(a_2i_k)f(\xi i_k).
\end{aligned}$$

### The Integrals $I_2^{(\xi)}(i_k)$

Having made the usual substitutions,

$$x_1 \longmapsto x_1 + \pi^{-Z_2}c(\pi^{Z_2}x_2 + \ldots + \pi^k i_k) \quad x_2 \longmapsto \pi^{Z_2}x_2 + \pi^{Z_2-1}i_{Z_2-1} + \ldots + \pi^k i_k$$
$$\Rightarrow \mathrm{d}\mathcal{X} \longmapsto \rho^{-Z_2}\mathrm{d}\mathcal{X} \equiv \mathrm{d}\mathcal{X} \pmod{m},$$

we find that the integrals $I_2^{(\xi)}(i_k)$ satisfy,

$$I_2^{(\xi)}(i_k) = \int_{\mathfrak{O}_\nu^2} f\left(\genfrac{}{}{0pt}{}{\pi^{X_1}a_1x_1+\pi^{X_1-Z_2}a_1c_2(\pi^{Z_2}x_2+\ldots+\pi^k i_k)+\pi^{X_2-Z_1}a_2c_1(\pi^{Z_2}x_2+\ldots+\pi^k i_k)}{\pi^{X_2}a_2(\pi^{Z_2}x_2+\ldots+\pi^k i_k)}\right) f(\xi c_2 i_k)\mathrm{d}\mathcal{X}.$$

Before we begin let us note that we already have,

$$X_1 > X_1 - (Z_2 - k) \quad \text{and} \quad X_2 + k > X_2 - Z_1 + k.$$

Therefore whenever $X_1 - X_2 \neq Z_2 - Z_1$ the function $f$ satisfies,

$$f\left(\genfrac{}{}{0pt}{}{\pi^{X_1}a_1(x_1+\pi^{-Z_2}c_2(\ldots+\pi^k i_k))+\pi^{X_2-Z_1}a_2c_1(\ldots+\pi^k i_k)}{\pi^{X_2}a_2(\pi^{Z_2}x_2+\ldots+\pi^k i_k)}\right) = f(\pi^{X_1-Z_2+k}a_1c_2i_k + \pi^{X_2-Z_1+k}a_2c_1i_k).$$

Using this fact we immediately see that, unlike the previous theorems, we must begin by splitting the integrals $I_2^{(\xi)}(i_k)$ into three distinct cases. We shall start by considering the following two simple cases.

**Case (1) : $X_1 - X_2 < Z_2 - Z_1$**

Since $X_1 - X_2 \neq Z_2 - Z_1$ we simply find,

$$\sum_{i_k\in\Pi\backslash 0} I_2^{(\xi)}(i_k) = \sum_{i_k\in\Pi\backslash 0} f(a_1c_2i_k)f(\xi c_2 i_k).$$

**Case (2) : $X_1 - X_2 > Z_2 - Z_1$**

Once again we simply find,

$$\sum_{i_k\in\Pi\backslash 0} I_2^{(\xi)}(i_k) = \sum_{i_k\in\Pi\backslash 0} f(a_2c_1i_k)f(\xi c_2 i_k).$$

**Case (3) : $X_1 - X_2 = Z_2 - Z_1$**

Now let us suppose that $X_1 - X_2 = Z_2 - Z_1$. In this case we are able to write,

$$I_2^{(\xi)}(i_k) = f(\xi c_2 i_k)\int_{\mathfrak{O}_\nu^2} f\left(\genfrac{}{}{0pt}{}{\pi^{X_1}a_1x_1+\pi^{X_2-Z_1}(a_1c_2+c_1a_2)(\pi^{Z_2}x_2+\ldots+\pi^k i_k))}{\pi^{X_2+k}a_2i_k}\right)\mathrm{d}\mathcal{X}. \tag{5.1}$$

However, in order to calculate this integral we shall need to split this up into a further four separate cases.

**Case (3.1)** $:$ $(a_1c_2 + c_1a_2) = 0$

To begin with we shall consider the case when $(a_1c_2 + c_1a_2) = 0$. Re-arranging (5.1) we quickly discover,

$$I_2^{(\xi)}(i_k) = f(\xi c_2 i_k)\int_{\mathfrak{O}_\nu^2} f\left(\tfrac{\pi^{X_1}a_1x_1}{\pi^{X_2+k}a_2i_k}\right)\mathrm{d}\mathcal{X}$$
$$= \begin{cases} f(a_2i_k)f(\xi c_2i_k) & (X_1 - X_2) > k \\ f(a_2i_k)f(\xi c_2i_k) + f(\xi c_2i_k)(\rho^{k-(X_1-X_2)+1} - 1)/m & (X_1 - X_2) \le k. \end{cases}$$

This gives us a further two cases,

**(3.1.1)** $(X_1 - X_2) > k :$

$$\sum_{i_k\in\Pi\backslash 0} I_2^{(\xi)}(i_k) = \sum_{i_k\in\Pi\backslash 0} f(a_2i_k)f(\xi c_2i_k),$$

**(3.1.2)** $(X_1 - X_2) \le k :$

$$\sum_{i_k\in\Pi\backslash 0} I_2^{(\xi)}(i_k) = \sum_{i_k\in\Pi\backslash 0} f(a_2i_k)f(\xi c_2i_k) + \frac{(\rho-1)}{m}(k - (X_1 - X_2) + 1).$$

In the remaining cases we shall no longer have $(a_1c_2 + c_1a_2) = 0$. Therefore, in order to continue we first define

$$(a_1c_2 + c_1a_2) := \pi^D d \quad \text{for some } D \ge 0 \text{ and where } |d|_\nu = 1.$$

**Case (3.2)** $:$ $X_1 - X_2 > k$

Referring to equation (5.1) we see that in this case we simply find,

$$I_2^{(\xi)}(i_k) = f(\xi c_2 i_k)\int_{\mathfrak{O}_\nu^2} f\left(\tfrac{\pi^{X_2-Z_1+k+D}di_k}{\pi^{X_2+k}a_2i_k}\right)\mathrm{d}\mathcal{X}$$
$$= \begin{cases} f(di_k)f(\xi c_2i_k) & D \le Z_1 \\ f(a_2i_k)f(\xi c_2i_k) & D > Z_1 \end{cases}$$

Therefore, whenever $X_1 - X_2 > k$, we shall again have two distinct results.

**(3.2.1)** $D \le Z_1 \Rightarrow (a_1c_2 + c_1a_2) \not\equiv 0(\pi^{Z_1+1}) :$

$$\sum_{i_k\in\Pi\backslash 0} I_2^{(\xi)}(i_k) = \sum_{i_k\in\Pi\backslash 0} f(di_k)f(\xi c_2i_k),$$

**(3.2.2)** $D > Z_1 \Rightarrow (a_1c_2 + c_1a_2) \equiv 0(\pi^{Z_1+1}) :$

$$\sum_{i_k\in\Pi\backslash 0} I_2^{(\xi)}(i_k) = \sum_{i_k\in\Pi\backslash 0} f(a_2i_k)f(\xi c_2i_k).$$

**Case (3.3)** : $X_1 - X_2 \le k$

In this case equation (5.1) becomes,

$$I_2^{(\xi)}(i_k) = f(\xi c_2 i_k) \int_{\mathfrak{O}_\nu^2} f\Big(\frac{\pi^{X_1} a_1 x_1 + \pi^{X_1 + D - (Z_2 - k)}(di_k + O(\pi))}{\pi^{X_2 + k} a_2 i_k}\Big) \mathrm{d}\mathcal{X}.$$

In order to calculate this integral we find that we must consider a further three possible cases.

**Case (3.3.1)** : $D < Z_2 - k$

In this the simplest case we see that whenever $(a_1 c_2 + c_1 a_2) \not\equiv 0\ (\pi^{Z_2 - k})$ the integral satisfies,

$$\sum_{i_k \in \Pi \backslash 0} I_2^{(\xi)}(i_k) = \sum_{i_k \in \Pi \backslash 0} f(di_k) f(\xi c_2 i_k).$$

**Case (3.3.2)** : $Z_2 - k \le D \le Z_1$

Once again returning to equation (5.1) we find that, since $Z_2 - k \le D \le Z_1$, we are able to express the integral as,

$$\begin{aligned} I_2^{(\xi)}(i_k) = f(\xi c_2 i_k) & \int_{\mathfrak{O}_\nu \backslash \pi^{D-(Z_2-k)} \mathfrak{O}_\nu} f(a_1 x_1) \mathrm{d}x_1 \\ & + \int_{\mathfrak{O}_\nu} \int_{\pi^{D-(Z_2-k)} \mathfrak{O}_\nu \backslash \pi^{D-(Z_2-k)+1} \mathfrak{O}_\nu} f\Big(\frac{\pi^{X_1} a_1 x_1 + \pi^{X_1 + D - (Z_2 - k)}(di_k + O(\pi))}{\pi^{X_2 + k} a_2 i_k}\Big) f(\xi c_2 i_k) \mathrm{d}\mathcal{X} \\ & + f(di_k) f(\xi c_2 i_k) \int_{\mathfrak{O}_\nu} \int_{\pi^{D-(Z_2-k)+1} \mathfrak{O}_\nu} \mathrm{d}\mathcal{X}. \end{aligned}$$

Since the first and last terms in this expression are easily calculated it remains only to find the integral,

$$\tilde{I}_2^{(\xi)}(i_k) = f(\xi c_2 i_k) \int_{\mathfrak{O}_\nu} \int_{\pi^{D-(Z_2-k)} \mathfrak{O}_\nu \backslash \pi^{D-(Z_2-k)+1} \mathfrak{O}_\nu} f\Big(\frac{\pi^{X_1} a_1 x_1 + \pi^{X_1 + D - (Z_2 - k)}(di_k + O(\pi))}{\pi^{X_2 + k} a_2 i_k}\Big) \mathrm{d}\mathcal{X}.$$

However, having made the substitution $x_1 \longmapsto \pi^{D-(Z_2-k)} x_1$ we once again find,

$$\begin{aligned} \tilde{I}_2^{(\xi)}(i_k) &= f(\xi c_2 i_k) \int_{\mathfrak{O}_\nu} \int_{\substack{x_1 \in \mathfrak{O}_\nu^\times \\ a_1 x_1 + di_k \not\equiv 0(\pi)}} f(a_1 x_1 + di_k) \mathrm{d}\mathcal{X} + f(\xi c_2 i_k) \int_{\mathfrak{O}_\nu} \int_{\substack{x_1 \in \mathfrak{O}_\nu^\times \\ a_1 x_1 + di_k \equiv 0(\pi)}} f\Big(\frac{(a_1 x_1 + di_k) + \pi di_{k+1} + \dots}{\pi^{Z_1 - D} a_2 i_k}\Big) \mathrm{d}\mathcal{X} \\ &= K_2^{(\xi)}(i_k) + K_3^{(\xi)}(i_k). \end{aligned}$$

Finally, using the results given in Lemma 5.5.2 and Lemma 5.5.3, we may conclude that whenever $(a_1c_2 + c_1a_2) \equiv 0(\pi^{Z_2-k})$ with $(a_1c_2 + c_1a_2) \not\equiv 0(\pi^{Z_1+1})$ we have,

$$\begin{aligned}\sum_{i_k\in\Pi\backslash 0} I_2^{(\xi)}(i_k) &= \frac{(\rho-1)}{m}(D-(Z_2-k)) \\ &\quad + \Bigg(\frac{(\rho-1)}{m} - \sum_{i_k\in\Pi\backslash 0} f(di_k)f(\xi c_2 i_k) + \frac{(\rho-1)}{m}(Z_1-D) \\ &\quad + \sum_{i_k\in\Pi\backslash 0} f(a_2i_k)f(\xi c_2 i_k)\Bigg) + \sum_{i_k\in\Pi\backslash 0} f(di_k)f(\xi c_2 i_k) \\ &= \frac{(\rho-1)}{m}((Z_1-Z_2)+(k+1)) + \sum_{i_k\in\Pi\backslash 0} f(a_2i_k)f(\xi c_2 i_k).\end{aligned}$$

**Case (3.3.3)** : $Z_1 < D$

Finally, by considering equation (5.1) in the case when $D > Z_1$ we find that,

$$\begin{aligned}I_2^{(\xi)}(i_k) &= \int_{\mathfrak{O}_\nu^2} f\big(\tfrac{\pi^{X_1}a_1x_1}{\pi^{X_2+k}a_2i_k}\big)f(\xi c_2 i_k)\mathrm{d}\mathcal{X} \\ &= f(\xi c_2 i_k)\int_{\mathfrak{O}_\nu\backslash\pi^{k-(X_1-X_2)+1}\mathfrak{O}_\nu} f(a_1x_1)\mathrm{d}x_1 \; + \; f(a_2i_k)f(\xi c_2 i_k)\int_{\pi^{k-(X_1-X_2)+1}\mathfrak{O}_\nu}\mathrm{d}x_1 \\ &= \frac{1}{m}(\rho^{k-(X_1-X_2)+1}-1)f(\xi c_2 i_k) \; + \; f(a_2i_k)f(\xi c_2 i_k).\end{aligned}$$

Therefore, whenever $(a_1c_2 + c_1a_2) \equiv 0(\pi^{Z_1+1})$, we have

$$\sum_{i_k\in\Pi\backslash 0} I_2^{(\xi)}(i_k) \equiv \frac{(\rho-1)}{m}(k-(X_1-X_2)+1) \; + \; \sum_{i_k\in\Pi\backslash 0} f(a_2i_k)f(\xi c_2 i_k).$$

Having calculated all of the integrals $I_1^{(\xi)}(i_k)$ and $I_2^{(\xi)}(i_k)$, we are now in a position to consider each of the possible cases which arise when evaluating the cocycle $dec_\nu$. Let us note that, since we have already seen similar calculations throughout this chapter, we shall begin to omit some of the detail.

Results for $(X_1 - X_2) + Z_1 < 0$ :

| | $0 \leq k < Z_2$ |
|---|---|
| $I_1^{(\xi)}(i_k)$ | Case (iii) |
| $I_2^{(\xi)}(i_k)$ | Case (1) |

$$\begin{aligned} dec_\nu(n_1\alpha_2, n_2) &= \prod_{k=0}^{Z_2-1} \prod_{\xi\in\mu_m} \xi^{\sum_{i_k} I^{(\xi)}(i_k)} \prod_{k=0}^{Z_2-1} \prod_{\xi\in\mu_m} \xi^{-\sum_{i_k} I_2^{(\xi)}(i_k)} \\ &= (-1)^{\frac{(\rho-1)}{m} Z_2(X_1-X_2)} \, (a_2,\pi)_{\nu,m}^{Z_2} \cdot (a_1,\pi)_{\nu,m}^{-Z_2} \\ &= (a_2/a_1,\pi)_{\nu,m}^{Z_2} \quad (-1)^{\frac{(\rho-1)}{2^r} Z_2(X_1-X_2)}. \end{aligned}$$

Results for $0 \leq (X_1 - X_2) + Z_1 < Z_2$ :

| | $0 \leq k < (X_1 - X_2) + Z_1$ | $(X_1 - X_2) + Z_1 \leq k < Z_2$ |
|---|---|---|
| $I_1^{(\xi)}(i_k)$ | Case (i) | Case (ii) |
| $I_2^{(\xi)}(i_k)$ | Case (1) | |

$$\begin{aligned} dec_\nu(n_1\alpha_2, n_2) = &\prod_{k=0}^{(X_1-X_2)+Z_1-1} \cdot \prod_{k=(X_1-X_2)+Z_1}^{Z_2-1} \left( \prod_{\xi\in\mu_m} \xi^{\sum_{i_k} I^{(\xi)}(i_k)} \right) \prod_{k=0}^{Z_2-1} \left( \prod_{\xi\in\mu_m} \xi^{-\sum_{i_k} I_2^{(\xi)}(i_k)} \right) \\ = &(-1)^{\frac{(\rho-1)}{m} \frac{((X_1-X_2)+Z_1)((X_1-X_2)+Z_1+1)}{2}} (c_1a_2,\pi)_{\nu,m}^{(X_1-X_2)+Z_1} \\ &.(-1)^{\frac{(\rho-1)}{m}((X_1-X_2)+1)(Z_2-((X_1-X_2)+Z_1))} (-a_1/c_1,\pi)_{\nu,m}^{Z_2-((X_1-X_2)+Z_1)} \\ &(c_1a_2,\pi)_{\nu,m}^{Z_2-((X_1-X_2)+Z_1)} (a_1,\pi)_{\nu,m}^{Z_2-((X_1-X_2)+Z_1)} \cdot (a_1,\pi)_{\nu,m}^{-Z_2} \\ = &(a_2/a_1,\pi)_{\nu,m}^{Z_2} \quad (c_1,\pi)_{\nu,m}^{(X_1-X_2)+Z_1} \\ &(-1)^{\frac{(\rho-1)}{2^r} Z_2(X_1-X_2)} \quad (-1)^{\frac{(\rho-1)}{2^r} \frac{(X_1-X_2)((X_1-X_2)-1)}{2}} \quad (-1)^{\frac{(\rho-1)}{2^r} \frac{Z_1(Z_1+1)}{2}}. \end{aligned}$$

Results for $Z_2 < (X_1 - X_2) + Z_1$ :

| | $0 \leq k < Z_2$ |
|---|---|
| $I_1^{(\xi)}(i_k)$ | Case (i) |
| $I_2^{(\xi)}(i_k)$ | Case (2) |

$$\begin{aligned} dec_\nu(n_1\alpha_2, n_2) &= \prod_{k=0}^{Z_2-1} \prod_{\xi\in\mu_m} \xi^{\sum_{i_k} I^{(\xi)}(i_k)} \prod_{k=0}^{Z_2-1} \prod_{\xi\in\mu_m} \xi^{-\sum_{i_k} I_2^{(\xi)}(i_k)} \\ &= (-1)^{\frac{(\rho-1)}{m} \frac{Z_2(Z_2+1)}{2}} (c_1a_2,\pi)_{\nu,m}^{Z_2} \cdot (a_2c_1/c_2,\pi)_{\nu,m}^{-Z_2} \\ &= (c_2,\pi)_{\nu,m}^{Z_2} (-1)^{\frac{(\rho-1)}{2^r} \frac{Z_2(Z_2+1)}{2}}. \end{aligned}$$

**Results for** $X_1 - X_2 = Z_2 - Z_1, \quad (a_1c_2 + c_1a_2) = 0:$

Taking a moment to consider this case we see that, since we have $X_1 - X_2 = Z_2 - Z_1 \le Z_2$, we shall in fact have three distinct results in this section. For each of these results, the integrals $I_1^{(\xi)}(i_k)$ shall be given by Case (i). When considering the integrals $I_2^{(\xi)}(i_k)$ we shall require the various results given in Case (3.1).

- $X_1 - X_2 = Z_2 - Z_1 \le 0:$

| | $0 \le k < Z_2$ |
|---|---|
| $I_1^{(\xi)}(i_k)$ | Case (i) |
| $I_2^{(\xi)}(i_k)$ | Case (3.1.2) |

$$\begin{aligned} dec_\nu(n_1\alpha_2, n_2) &= (-1)^{\frac{(\rho-1)}{m}\frac{Z_2(Z_2+1)}{2}}(c_1a_2,\pi)_{\nu,m}^{Z_2} \cdot (-1)^{\frac{(\rho-1)}{m}\frac{Z_2(Z_2+1)}{2}}(-1)^{\frac{(\rho-1)}{m}Z_2(X_1-X_2)}(a_2/c_2,\pi)_{\nu,m}^{-Z_2} \\ &= (c_1c_2,\pi)_{\nu,m}^{Z_2} \quad (-1)^{\frac{(\rho-1)}{2^r}Z_2}(-1)^{\frac{(\rho-1)}{2^r}Z_1Z_2}. \end{aligned}$$

- $0 < X_1 - X_2 = Z_2 - Z_1 < Z_2:$

| | $0 \le k < (X_1 - X_2)$ | $(X_1 - X_2) \le k < Z_2$ |
|---|---|---|
| $I_1^{(\xi)}(i_k)$ | Case (i) | |
| $I_2^{(\xi)}(i_k)$ | Case (3.1.1) | Case (3.1.2) |

$$\begin{aligned} dec_\nu(n_1\alpha_2, n_2) &= (-1)^{\frac{(\rho-1)}{m}\frac{Z_2(Z_2+1)}{2}}(c_1a_2,\pi)_{\nu,m}^{Z_2} \cdot (-1)^{\frac{(\rho-1)}{m}\frac{(Z_2-(X_1-X_2))(Z_2-(X_1-X_2)+1)}{2}}(a_2/c_2,\pi)_{\nu,m}^{-Z_2} \\ &= (c_1c_2,\pi)_{\nu,m}^{Z_2} \quad (-1)^{\frac{(\rho-1)}{2^r}\frac{Z_2(Z_2+1)}{2}}(-1)^{\frac{(\rho-1)}{2^r}\frac{Z_1(Z_1+1)}{2}}. \end{aligned}$$

- $0 < X_1 - X_2 = Z_2, \quad Z_1 = 0:$

| | $0 \le k < Z_2$ |
|---|---|
| $I_1^{(\xi)}(i_k)$ | Case (i) |
| $I_2^{(\xi)}(i_k)$ | Case (3.1.1) |

$$\begin{aligned} dec_\nu(n_1\alpha_2, n_2) &= (-1)^{\frac{(\rho-1)}{m}\frac{Z_2(Z_2+1)}{2}}(c_1a_2,\pi)_{\nu,m}^{Z_2} \cdot (a_2/c_2,\pi)_{\nu,m}^{-Z_2} \\ &= (c_1c_2,\pi)_{\nu,m}^{Z_2} \quad (-1)^{\frac{(\rho-1)}{2^r}\frac{Z_2(Z_2+1)}{2}}. \end{aligned}$$

**Results for** $X_1 - X_2 = Z_2 - Z_1, \quad (a_1c_2 + c_1a_2) \neq 0:$

Let us remind ourselves that whenever $(a_1c_2 + c_1a_2) \neq 0$ we have defined,

$$(a_1c_2 + c_1a_2) := \pi^D d \quad \text{for some } D \geq 0 \text{ and where } \quad |d|_\nu = 1.$$

Once again, for each of the results in this section the integrals $I_1^{(\xi)}(i_k)$ shall be given by Case (i). When considering the integrals $I_2^{(\xi)}(i_k)$ we see that we no longer have just three cases depending on the value of $X_1 - X_2 = Z_2 - Z_1$. That is, in this section each of these cases will further split depending on the value of the exponent $D$.

- $X_1 - X_2 = Z_2 - Z_1 \leq 0:$

Throughout this section we have $X_1 - X_2 \leq k$ for all $0 \leq k < Z_2$. Therefore, for the integrals $I_2^{(\xi)}(i_k)$, we shall require the results given in Case (3.3). Now let us consider the value of the integer $D \geq 0$.

- $D = 0:$

| | $0 \leq k < Z_2$ |
|---|---|
| $I_1^{(\xi)}(i_k)$ | Case (i) |
| $I_2^{(\xi)}(i_k)$ | Case (3.3.1) |

$$\begin{aligned} dec_\nu(n_1\alpha_2, n_2) &= (-1)^{\frac{(\rho-1)}{m}\frac{Z_2(Z_2+1)}{2}} (c_1a_2, \pi)_{\nu,m}^{Z_2} \cdot (d/c_2, \pi)_{\nu,m}^{-Z_2} \\ &= (c_1c_2a_2/d, \pi)_{\nu,m}^{Z_2} \quad (-1)^{\frac{(\rho-1)}{2^r}\frac{Z_2(Z_2+1)}{2}}. \end{aligned}$$

- $1 \leq D < Z_2:$

| | $0 \leq k < Z_2 - D$ | $Z_2 - D \leq k < Z_2$ |
|---|---|---|
| $I_1^{(\xi)}(i_k)$ | Case (i) | |
| $I_2^{(\xi)}(i_k)$ | Case (3.3.1) | Case (3.3.2) |

$$\begin{aligned} dec_\nu(n_1\alpha_2, n_2) &= (-1)^{\frac{(\rho-1)}{m}\frac{Z_2(Z_2+1)}{2}} (c_1a_2, \pi)_{\nu,m}^{Z_2} \cdot (d/c_2, \pi)_{\nu,m}^{D-Z_2} \cdot \\ &\quad (-1)^{\frac{(\rho-1)}{m}(Z_1-D)D} (-1)^{\frac{(\rho-1)}{m}\frac{D(D+1)}{2}} (a_2/c_2, \pi)_{\nu,m}^{-D} \\ &= (c_1c_2a_2/d, \pi)_{\nu,m}^{Z_2} \quad (d/a_2, \pi)_{\nu,m}^{D} \\ &\quad (-1)^{\frac{(\rho-1)}{2^r}\frac{Z_2(Z_2+1)}{2}} (-1)^{\frac{(\rho-1)}{2^r}\frac{D(D-1)}{2}} (-1)^{\frac{(\rho-1)}{2^r}Z_1D}. \end{aligned}$$

∘ $Z_2 \le D \le Z_1$ :

| | $0 \le k < Z_2$ |
|---|---|
| $I_1^{(\xi)}(i_k)$ | Case (i) |
| $I_2^{(\xi)}(i_k)$ | Case (3.3.2) |

$$\begin{aligned} dec_\nu(n_1\alpha_2, n_2) &= (-1)^{\frac{(\rho-1)}{m}\frac{Z_2(Z_2+1)}{2}}(c_1a_2,\pi)_{\nu,m}^{Z_2}\cdot(-1)^{\frac{(\rho-1)}{m}(Z_1-Z_2)Z_2}(-1)^{\frac{(\rho-1)}{m}\frac{Z_2(Z_2+1)}{2}}(a_2/c_2,\pi)_{\nu,m}^{-Z_2} \\ &= (c_1c_2,\pi)_{\nu,m}^{Z_2} \quad (-1)^{\frac{(\rho-1)}{2^r}(Z_1-1)Z_2}. \end{aligned}$$

∘ $Z_1 < D$ :

| | $0 \le k < Z_2$ |
|---|---|
| $I_1^{(\xi)}(i_k)$ | Case (i) |
| $I_2^{(\xi)}(i_k)$ | Case (3.3.3) |

$$\begin{aligned} dec_\nu(n_1\alpha_2, n_2) &= (-1)^{\frac{(\rho-1)}{m}\frac{Z_2(Z_2+1)}{2}}(c_1a_2,\pi)_{\nu,m}^{Z_2}\cdot(-1)^{\frac{(\rho-1)}{m}(X_1-X_2)Z_2}(-1)^{\frac{(\rho-1)}{m}\frac{Z_2(Z_2+1)}{2}}(a_2/c_2,\pi)_{\nu,m}^{-Z_2} \\ &= (c_1c_2,\pi)_{\nu,m}^{Z_2} \quad (-1)^{\frac{(\rho-1)}{2^r}(X_1-X_2)Z_2}. \end{aligned}$$

- $0 < X_1 - X_2 = Z_2 - Z_1 < Z_2$ :

For this more complicated case we can no longer just consider $X_1 - X_2 \le k$. That is, in this section we shall have

$$k < X_1 - X_2 \quad \text{for } 0 \le k < Z_2 - Z_1, \qquad \text{and} \qquad X_1 - X_2 \le k \quad \text{for } Z_2 - Z_1 \le k < Z_2.$$

Therefore, for the integrals $I_2^{(\xi)}(i_k)$, we must consider the results given in both Cases (3.2) and (3.3). This will once again give us four possible results depending on the four possible values of the integer $D \ge 0$.

∘ $D = 0$ :

| | $0 \le k < Z_2 - Z_1$ | $Z_2 - Z_1 \le k < Z_2$ |
|---|---|---|
| $I_1^{(\xi)}(i_k)$ | Case (i) | |
| $I_2^{(\xi)}(i_k)$ | Case (3.2.1) | Case (3.3.1) |

$$\begin{aligned} dec_\nu(n_1\alpha_2, n_2) &= (-1)^{\frac{(\rho-1)}{m}\frac{Z_2(Z_2+1)}{2}}(c_1a_2,\pi)_{\nu,m}^{Z_2}\cdot(d/c_2,\pi)_{\nu,m}^{-(X_1-X_2)} \\ &\quad (d/c_2,\pi)_{\nu,m}^{-(Z_2-(X_1-X_2))} \\ &= (c_1c_2a_2/d,\pi)_{\nu,m}^{Z_2} \quad (-1)^{\frac{(\rho-1)}{2^r}\frac{Z_2(Z_2+1)}{2}}. \end{aligned}$$

∘ $1 \leq D < Z_1$ :

| | $0 \leq k < Z_2 - Z_1$ | $Z_2 - Z_1 \leq k < Z_2 - D$ | $Z_2 - D \leq k < Z_2$ |
|---|---|---|---|
| $I_1^{(\xi)}(i_k)$ | Case (i) | | |
| $I_2^{(\xi)}(i_k)$ | Case (3.2.1) | Case (3.3.1) | Case (3.3.2) |

$$\begin{aligned} dec_\nu(n_1\alpha_2, n_2) &= (-1)^{\frac{(\rho-1)}{m}\frac{Z_2(Z_2+1)}{2}}(c_1a_2,\pi)_{\nu,m}^{Z_2} \cdot (d/c_2,\pi)_{\nu,m}^{-(X_1-X_2)} \\ &\quad (d/c_2,\pi)_{\nu,m}^{D-Z_2+(X_1-X_2)} \cdot (-1)^{\frac{(\rho-1)}{m}(Z_1-D)D}(-1)^{\frac{(\rho-1)}{m}\frac{D(D+1)}{2}}(a_2/c_2,\pi)_{\nu,m}^{-D} \\ &= (c_1c_2a_2/d,\pi)_{\nu,m}^{Z_2} \quad (d/a_2,\pi)_{\nu,m}^{D} \\ &\quad (-1)^{\frac{(\rho-1)}{2^r}\frac{Z_2(Z_2+1)}{2}}(-1)^{\frac{(\rho-1)}{2^r}\frac{D(D-1)}{2}}(-1)^{\frac{(\rho-1)}{2^r}Z_1D}. \end{aligned}$$

∘ $D = Z_1$ :

| | $0 \leq k < Z_2 - Z_1$ | $Z_2 - Z_1 \leq k < Z_2$ |
|---|---|---|
| $I_1^{(\xi)}(i_k)$ | Case (i) | |
| $I_2^{(\xi)}(i_k)$ | Case (3.2.1) | Case (3.3.2) |

$$\begin{aligned} dec_\nu(n_1\alpha_2, n_2) &= (-1)^{\frac{(\rho-1)}{m}\frac{Z_2(Z_2+1)}{2}}(c_1a_2,\pi)_{\nu,m}^{Z_2} \cdot (d/c_2,\pi)_{\nu,m}^{-(X_1-X_2)} \\ &\quad (-1)^{\frac{(\rho-1)}{m}\frac{Z_1(Z_1+1)}{2}}(a_2/c_2,\pi)_{\nu,m}^{-Z_1} \\ &= (c_1c_2,\pi)_{\nu,m}^{Z_2} \quad (a_2/d,\pi)_{\nu,m}^{(X_1-X_2)} \\ &\quad (-1)^{\frac{(\rho-1)}{2^r}\frac{Z_2(Z_2+1)}{2}}(-1)^{\frac{(\rho-1)}{2^r}\frac{Z_1(Z_1+1)}{2}}. \end{aligned}$$

∘ $Z_1 < D$ :

| | $0 \leq k < Z_2 - Z_1$ | $Z_2 - Z_1 \leq k < Z_2$ |
|---|---|---|
| $I_1^{(\xi)}(i_k)$ | Case (i) | |
| $I_2^{(\xi)}(i_k)$ | Case (3.2.2) | Case (3.3.3) |

$$\begin{aligned} dec_\nu(n_1\alpha_2, n_2) &= (-1)^{\frac{(\rho-1)}{m}\frac{Z_2(Z_2+1)}{2}}(c_1a_2,\pi)_{\nu,m}^{Z_2} \cdot (a_2/c_2,\pi)_{\nu,m}^{-(X_1-X_2)} \\ &\quad \cdot (-1)^{\frac{(\rho-1)}{m}\frac{Z_1(Z_1+1)}{2}}(a_2/c_2,\pi)_{\nu,m}^{(X_1-X_2)-Z_2} \\ &= (c_1c_2,\pi)_{\nu,m}^{Z_2} \quad (-1)^{\frac{(\rho-1)}{2^r}\frac{Z_2(Z_2+1)}{2}}(-1)^{\frac{(\rho-1)}{2^r}\frac{Z_1(Z_1+1)}{2}}. \end{aligned}$$

- $0 < X_1 - X_2 = Z_2$ :

In this the simplest case we must have $Z_1 = 0$. Since we always have $k < X_1 - X_2$, for the integrals $I_2^{(\xi)}(i_k)$, we need only consider the results given in Case (3.2) where $Z_1$ has been replaced with 0. This leaves us with just two possibilities.

- $D = 0$ :

| | $0 \leq k < Z_2$ |
|---|---|
| $I_1^{(\xi)}(i_k)$ | Case (i) |
| $I_2^{(\xi)}(i_k)$ | Case (3.2.1) |

$$\begin{aligned} dec_\nu(n_1\alpha_2, n_2) &= (-1)^{\frac{(\rho-1)}{m}\frac{Z_2(Z_2+1)}{2}} (c_1 a_2, \pi)_{\nu,m}^{Z_2} \cdot (d/c_2, \pi)_{\nu,m}^{-Z_2} \\ &= (c_1 c_2 a_2/d, \pi)_{\nu,m}^{Z_2} \quad (-1)^{\frac{(\rho-1)}{2^r}\frac{Z_2(Z_2+1)}{2}}. \end{aligned}$$

- $D > 0$ :

| | $0 \leq k < Z_2$ |
|---|---|
| $I_1^{(\xi)}(i_k)$ | Case (i) |
| $I_2^{(\xi)}(i_k)$ | Case (3.2.2) |

$$\begin{aligned} dec_\nu(n_1\alpha_2, n_2) &= (-1)^{\frac{(\rho-1)}{m}\frac{Z_2(Z_2+1)}{2}} (c_1 a_2, \pi)_{\nu,m}^{Z_2} \cdot (a_2/c_2, \pi)_{\nu,m}^{-Z_2} \\ &= (c_1 c_2, \pi)_{\nu,m}^{Z_2} \quad (-1)^{\frac{(\rho-1)}{2^r}\frac{Z_2(Z_2+1)}{2}}. \end{aligned}$$

In order to complete this proof it simply remains to note that the results given in the theorem do indeed coincide with the results for the cocycle that we have calculated above.

$\square$

Once again, as an immediate consequence of the previous theorem we are able to deduce the following Corollary.

**Corollary 4** *As with the previous theorem we define,*

$$n_1 := n_{1,2}(\pi^{-Z_1}c_1), \quad n_2 := n_{1,2}(\pi^{-Z_2}c_2) \in N \quad \textit{with } Z_2 > 0.$$

*Then, having defined* $(c_1 + c_2) := \pi^D d$ *for some* $D \geq 0$ *and where* $d = 0$ *or* $|d|_\nu = 1$, *the cocycle* $dec_\nu$ *satisfies,*

$\bullet Z_1 < 0:$

$$dec_\nu(\begin{pmatrix} 1 & \pi^{-Z_1}c_1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \pi^{-Z_2}c_2 \\ 0 & 1 \end{pmatrix}) = 1.$$

$\bullet 0 \leq Z_1 < Z_2:$

$$dec_\nu(\begin{pmatrix} 1 & \pi^{-Z_1}c_1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \pi^{-Z_2}c_2 \\ 0 & 1 \end{pmatrix}) = (c_1, \pi)_{\nu,m}^{Z_1} (-1)^{\frac{(\rho-1)}{2^r}\frac{Z_1(Z_1+1)}{2}}.$$

$\bullet 0 < Z_2 < Z_1:$

$$dec_\nu(\begin{pmatrix} 1 & \pi^{-Z_1}c_1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \pi^{-Z_2}c_2 \\ 0 & 1 \end{pmatrix}) = (c_2, \pi)_{\nu,m}^{Z_2} (-1)^{\frac{(\rho-1)}{2^r}\frac{Z_2(Z_2+1)}{2}}.$$

$\bullet Z_1 = Z_2,\ D > Z_1 \quad or \quad (c_1 + c_2) = 0:$

$$dec_\nu(\begin{pmatrix} 1 & \pi^{-Z_1}c_1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \pi^{-Z_2}c_2 \\ 0 & 1 \end{pmatrix}) = (c_1c_2, \pi)_{\nu,m}^{Z_2}$$

$\bullet Z_1 = Z_2,\ D \leq Z_1:$

$$dec_\nu(\begin{pmatrix} 1 & \pi^{-Z_1}c_1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \pi^{-Z_2}c_2 \\ 0 & 1 \end{pmatrix}) = (c_1c_2/d, \pi)_{\nu,m}^{Z_2} \quad (d, \pi)_{\nu,m}^{D} \quad .$$
$$(-1)^{\frac{(\rho-1)}{2^r}\frac{Z_2(Z_2+1)}{2}} (-1)^{\frac{(\rho-1)}{2^r}\frac{D(D-1)}{2}} (-1)^{\frac{(\rho-1)}{2^r}Z_2D}.$$

## 5.6 $dec_\nu$ on $N.M \times N$

**Theorem 5.6.1** *Let* $\eta_\psi \in \mathfrak{M}$, $\alpha_2 \in T$ *and* $n_1 := n_{1,2}(\pi^{-Z_1}c_1)$, $n_2 := n_{1,2}(\pi^{-Z_2}c_2) \in N$ *be as previously defined. Then, whenever* $Z_1 < 0$, *the cocycle* $dec_\nu$ *satisfies,*

$$dec_\nu\left(\left(\begin{pmatrix} 1 & \pi^{-Z_1}c_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \pi^{X_1}a_1 & 0 \\ 0 & \pi^{X_2}a_2 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & \pi^{-Z_2}c_2 \\ 0 & 1 \end{pmatrix}\right)\right) = dec_\nu(\alpha_2\eta_\psi, n_2).$$

**PROOF OF THEOREM:**

Whenever $Z_1 < 0$ the matrix $n_1$ satisfies,

$$n_1 := \begin{pmatrix} 1 & \pi^{-Z_1}c_1 \\ 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2 \pmod{\pi}.$$

Therefore, by Lemma 1.4.4, we have $fn_1 = f$ and our result immediately follows. $\square$

**Theorem 5.6.2** *Let $\eta_\psi \in \mathfrak{M}$ and $\alpha_2 \in T$ be as previously defined. Also let,*

$$n_1 := n_{1,2}(\pi^{-Z_1}c_1),\ n_2 := n_{1,2}(\pi^{-Z_2}c_2) \in N,$$

*where $Z_1 \geq 0$, $Z_2 > 0$. Then, having defined*

$$(a_2c_1c_2 - a_1) := \pi^H h,$$

*for some $H \geq 0$ and where $h = 0$ or $|\, h\,|_\nu = 1$, the cocycle $dec_\nu$ satisfies,*

- $Z_2 < (X_2 - X_1) - Z_1$ :

$$dec_\nu(n_1\alpha_2\eta_\psi, n_2) = (c_2, \pi)^{Z_2}_{\nu,m}\,(-1)^{\frac{(\rho-1)}{2^r}\frac{Z_2(Z_2+1)}{2}}.$$

- $(X_2 - X_1) - Z_1 < 0$ :

$$dec_\nu(n_1\alpha_2\eta_\psi, n_2) = (a_1/a_2, \pi)^{Z_2}_{\nu,m}(c_1,\pi)^{-2Z_2}_{\nu,m}\,(-1)^{\frac{(\rho-1)}{2^r}Z_2(X_2-X_1)}.$$

- $0 \leq (X_2 - X_1) - Z_1 < Z_2$ :

$$\begin{aligned} dec_\nu(n_1\alpha_2\eta_\psi, n_2) &= (a_1/a_2, \pi)^{Z_2}_{\nu,m}(c_1,\pi)^{-2Z_2}_{\nu,m}\,(-1)^{\frac{(\rho-1)}{2^r}Z_2(X_1-X_2)} \\ &\quad (c_1,\pi)^{(X_2-X_1)-Z_1}_{\nu,m}\,(-1)^{\frac{(\rho-1)}{2^r}\frac{(X_2-X_1)((X_2-X_1)+1)}{2}}\,(-1)^{\frac{(\rho-1)}{2^r}\frac{Z_1(Z_1+1)}{2}}. \end{aligned}$$

- $X_2 - X_1 = Z_1 + Z_2$ :

  ◦ $(a_2c_1c_2 - a_1) = 0$ or $H \not\leq Z_1, Z_2$ :

$Z_2 \leq Z_1$ : $$dec_\nu(n_1\alpha_2\eta_\psi, n_2) = (a_1/a_2, \pi)^{Z_2}_{\nu,m}\,(-1)^{\frac{(\rho-1)}{2^r}Z_1Z_2}$$

$Z_1 \leq Z_2$ : $$dec_\nu(n_1\alpha_2\eta_\psi, n_2) = (a_1/a_2, \pi)^{Z_2}_{\nu,m}\,(-1)^{\frac{(\rho-1)}{2^r}\frac{Z_2(Z_2-1)}{2}}\,(-1)^{\frac{(\rho-1)}{2^r}\frac{Z_1(Z_1+1)}{2}}$$

  ◦ $(a_2c_1c_2 - a_1) \neq 0$, $H \leq Z_1, Z_2$ :

$$\begin{aligned} dec_\nu(n_1\alpha_2\eta_\psi, n_2) &= (a_1/a_2, \pi)^{Z_2}_{\nu,m}(a_2c_2/h, \pi)^{Z_2-H}_{\nu,m} \\ &\quad (-1)^{\frac{(\rho-1)}{2^r}\frac{Z_2(Z_2-1)}{2}}\,(-1)^{\frac{(\rho-1)}{2^r}\frac{H(H-1)}{2}}\,(-1)^{\frac{(\rho-1)}{2^r}Z_1H}. \end{aligned}$$

**PROOF OF THEOREM:**

As with each of the theorems in this chapter we shall begin by considering the integrals $I_1^{(\xi)}(i_k)$ and $I_2^{(\xi)}(i_k)$.

### The Integrals $I_1^{(\xi)}(i_k)$

Having applied the usual substitution we find that the integrals $I_1^{(\xi)}(i_k)$ are dependent only on $i_k$ and are given by,

$$I_1^{(\xi)}(i_k) = \int_{\mathfrak{O}_\nu^2} f\Big(\substack{\pi^{X_2-Z_1}a_2c_1x_1 - \pi^{X_1}a_1(\pi^{Z_2}x_2+\ldots+\pi^k i_k) \\ \pi^{X_2}a_2x_1}\Big) f\Big(\substack{\xi x_1 \\ \xi\pi^k i_k}\Big)\mathrm{d}\mathcal{X}. \qquad (5.2)$$

Now, in order to calculate this integral we shall need to consider three distinct cases.

(i) $0 \leq k < (X_2 - X_1) - Z_1$, (ii) $0 \leq (X_2 - X_1) - Z_1 \leq k$, (iii) $(X_2 - X_1) - Z_1 < 0 \leq k$.

**Case (i)** : $0 \leq k < (X_2 - X_1) - Z_1$

In this, the simplest case, we already know that $X_2 \geq X_2 - Z_1 > X_1 + k$. Therefore the function $f$ must satisfy,

$$f\left(\tfrac{\pi^{X_2-Z_1}a_2c_1x_1 - \pi^{X_1}a_1(\pi^{Z_2}x_2+\ldots+\pi^k i_k)}{\pi^{X_2}a_2x_1}\right) = f(-a_1 i_k).$$

Returning to equation (5.2) this allows us to simply write,

$$\begin{aligned}
I_1^{(\xi)}(i_k) &= \int_{\mathfrak{O}_\nu^2} f(-a_1 i_k) f\left(\tfrac{\xi x_1}{\xi\pi^k i_k}\right) \mathrm{d}\mathcal{X} \\
&= f(-a_1 i_k) \int_{\mathfrak{O}_\nu \setminus \pi^{k+1}\mathfrak{O}_\nu} f(\xi x_1) \mathrm{d}x_1 \ + \ f(-a_1 i_k) f(\xi i_k) \int_{\pi^{k+1}\mathfrak{O}_\nu} \mathrm{d}x_1 \\
&= f(-a_1 i_k) \frac{1}{m}(\rho^{k+1} - 1) + f(-a_1 i_k) f(\xi i_k),
\end{aligned}$$

and taking the sum over the $i_k$'s we find,

$$\sum_{i_k \in \Pi\setminus 0} I_1^{(\xi)}(i_k) \quad = \quad \frac{(\rho-1)}{m}(k+1) + \sum_{i_k \in \Pi\setminus 0} f(-a_1 i_k) f(\xi i_k).$$

**Case (ii)** : $0 \leq (X_2 - X_1) - Z_1 \leq k$

In this case, when considering equation (5.2), we find

$$\begin{aligned}
I_1^{(\xi)}(i_k) \equiv & \int_{\mathfrak{O}_\nu \setminus \pi^{k+Z_1-(X_2-X_1)}\mathfrak{O}_\nu} f(a_2 c_1 x_1) f(\xi x_1) \mathrm{d}x_1 \\
& + \int_{\mathfrak{O}_\nu} \int_{\pi^{k+Z_1-(X_2-X_1)}\mathfrak{O}_\nu \setminus \pi^{k+Z_1-(X_2-X_1)+1}\mathfrak{O}_\nu} f\left(\tfrac{\pi^{X_2-Z_1}a_2c_1x_1 - \pi^{X_1}a_1(\pi^{Z_2}x_2+\ldots+\pi^k i_k)}{\pi^{X_2}a_2x_1}\right) f(\xi x_1) \mathrm{d}\mathcal{X} \\
& + f(-a_1 i_k) \int_{\pi^{k+Z_1-(X_2-X_1)+1}\mathfrak{O}_\nu \setminus \pi^{k+1}\mathfrak{O}_\nu} f(\xi x_1) \mathrm{d}x_1 \\
& + f(-a_1 i_k) f(\xi i_k) \int_{\pi^{k+1}\mathfrak{O}_\nu} \mathrm{d}x_1.
\end{aligned}$$

Once again we may disregard the first term by Lemma 5.2.2 as it has no dependence on $i_k$. The third and fourth terms in this expression are easily calculated so in order to evaluate this integral we need only concentrate on calculating,

$$\tilde{I}_1^{(\xi)}(i_k) := \int_{\mathfrak{O}_\nu} \int_{\pi^{k+Z_1-(X_2-X_1)}\mathfrak{O}_\nu \setminus \pi^{k+Z_1-(X_2-X_1)+1}\mathfrak{O}_\nu} f\left(\tfrac{\pi^{X_2-Z_1}a_2c_1x_1 - \pi^{X_1}a_1(\pi^{Z_2}x_2+\ldots+\pi^k i_k)}{\pi^{X_2}a_2x_1}\right) f(\xi x_1) \mathrm{d}\mathcal{X}.$$

Having made the substitution,

$$x_1 \longmapsto \pi^{Z_1-(X_2-X_1)+k}x_1, \qquad dx_1 \longmapsto dx_1 \pmod{m},$$

we find that,

$$\begin{aligned}
\tilde{I}_1^{(\xi)}(i_k) &= \int_{\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu^\times} f\Big(\tfrac{\pi^{X_1+k}(c_1a_2x_1-a_1i_k-\pi a_1 i_{k+1}-\dots-\pi^{Z_2-k}a_1x_2)}{\pi^{k+X_1+Z_1}a_2x_1}\Big) f(\xi x_1)\mathrm{d}\mathcal{X} \\
&= \int_{\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu^\times} f\Big(\tfrac{(c_1a_2x_1-a_1i_k-\pi a_1 i_{k+1}-\dots-\pi^{Z_2-k}a_1x_2)}{\pi^{Z_1}a_2x_1}\Big) f(\xi x_1)\mathrm{d}\mathcal{X} \\
&= \underset{c_1a_2x_1-a_1i_k\not\equiv 0(\pi)}{\int_{\mathfrak{O}_\nu}\int_{x_1\in\mathfrak{O}_\nu^\times}} f(c_1a_2x_1-a_1i_k) f(\xi x_1)\mathrm{d}\mathcal{X} \\
&\quad + \underset{c_1a_2x_1-a_1i_k\equiv 0(\pi)}{\int_{\mathfrak{O}_\nu}\int_{x_1\in\mathfrak{O}_\nu^\times}} f\Big(\tfrac{(c_1a_2x_1-a_1i_k)-\pi a_1 i_{k+1}-\dots-\pi^{Z_2-k}a_1x_2}{\pi^{Z_1}a_2x_1}\Big) f(\xi x_1)\mathrm{d}\mathcal{X}.
\end{aligned}$$

However in this last integral we have the condition $c_1a_2x_1-a_1i_k\equiv 0 \pmod{\pi}$. Therefore we must have,

$$f(\xi x_1) = f(\xi(a_1c_1^{-1}a_2^{-1})i_k) \qquad \text{and} \qquad f(\pi^{Z_1}a_2x_1) = f(\pi^{Z_1}a_2(a_1c_1^{-1}a_2^{-1}i_k)).$$

This allows us to write,

$$\begin{aligned}
\tilde{I}_1^{(\xi)}(i_k) &= \underset{c_1a_2x_1-a_1i_k\not\equiv 0(\pi)}{\int_{\mathfrak{O}_\nu}\int_{x_1\in\mathfrak{O}_\nu^\times}} f(c_1a_2x_1-a_1i_k) f(\xi x_1)\mathrm{d}\mathcal{X} \\
&\quad + \underset{c_1a_2x_1-a_1i_k\equiv 0(\pi)}{\int_{\mathfrak{O}_\nu}\int_{x_1\in\mathfrak{O}_\nu^\times}} f\Big(\tfrac{(c_1a_2x_1-a_1i_k)-\pi a_1 i_{k+1}-\dots-\pi^{Z_2-k}a_1x_2}{\pi^{Z_1}a_2(a_1c_1^{-1}a_2^{-1}i_k)}\Big) f(\xi(a_1c_1^{-1}a_2^{-1}i_k))\mathrm{d}\mathcal{X} \\
&= K_1^{(\xi)}(i_k) + K_3^{(\xi)}(i_k),
\end{aligned}$$

where the integrals $K_1^{(\xi)}(i_k)$ and $K_3^{(\xi)}(i_k)$ are those described in Lemmas 5.5.1 and 5.5.3 respectively.

Using the results given in Lemmas 5.5.1 and 5.5.3 we are able to conclude that,

$$\begin{aligned}
\sum_{i_k\in\Pi\backslash 0} \tilde{I}_1^{(\xi)}(i_k) &= \sum_{i_k\in\Pi\backslash 0} K_1^{(\xi)}(i_k) + \sum_{i_k\in\Pi\backslash 0} K_3^{(\xi)}(i_k) \\
&= \Big(\frac{(\rho-1)}{m} - \sum_{x_1\in\Pi\backslash 0} f(a_2c_1x_1)f(\xi x_1)\Big) \\
&\quad + \Big(\frac{(\rho-1)}{m}Z_1 + \sum_{i_k\in\Pi\backslash 0} f(a_2(a_1c_1^{-1}a_2^{-1})i_k)f(\xi(a_1c_1^{-1}a_2^{-1})i_k)\Big).
\end{aligned}$$

Returning to our calculation of the integrals $I_1^{(\xi)}(i_k)$ we find that whenever $0 \le (X_2 - X_1) - Z_1 \le k$ these integrals satisfy,

$$\begin{aligned}
\sum_{i_k \in \Pi\backslash 0} I_1^{(\xi)}(i_k) &= \Big(\frac{(\rho-1)}{m}(Z_1+1) - \sum_{x_1\in\Pi\backslash 0} f(a_2c_1x_1)f(\xi x_1) \\
&\qquad + \sum_{i_k\in\Pi\backslash 0} f(a_2(a_1c_1^{-1}a_2^{-1})i_k)f(\xi(a_1c_1^{-1}a_2^{-1})i_k)\Big) \\
&\qquad\qquad + \Big(\frac{(\rho-1)}{m}(Z_1-(X_2-X_1)) + \sum_{i_k\in\Pi\backslash 0} f(-a_1i_k)f(\xi i_k)\Big) \\
&= \frac{(\rho-1)}{m}((X_2-X_1)+1) - \sum_{x_1\in\Pi\backslash 0} f(a_2c_1x_1)f(\xi x_1) \\
&\qquad + \sum_{i_k\in\Pi\backslash 0} f(a_2(a_1c_1^{-1}a_2^{-1})i_k)f(\xi(a_1c_1^{-1}a_2^{-1})i_k) + \sum_{i_k\in\Pi\backslash 0} f(-a_1i_k)f(\xi i_k).
\end{aligned}$$

**Case (iii) : $(X_2 - X_1) - Z_1 < 0 \le k$**

Returning to the original problem, in this case equation (5.2) becomes,

$$\begin{aligned}
I_1^{(\xi)}(i_k) &= \int_{\mathfrak{O}_\nu\backslash\pi^{k+1}\mathfrak{O}_\nu} f(c_1a_2x_1)f(\xi x_1)\mathrm{d}x_1 \\
&\quad + f(\xi i_k)\int_{\pi^{k+1}\mathfrak{O}_\nu\backslash\pi^{k+Z_1-(X_2-X_1)}\mathfrak{O}_\nu} f(c_1a_2x_1)\mathrm{d}x_1 \\
&\quad + f(\xi i_k)\int_{\pi^{k+Z_1-(X_2-X_1)}\mathfrak{O}_\nu\backslash\pi^{k+Z_1-(X_2-X_1)+1}\mathfrak{O}_\nu} f\Big(\frac{\pi^{X_2-Z_1}a_2c_1x_1-\pi^{X_1}a_1(\pi^{Z_2}x_2+\ldots+\pi^k i_k)}{\pi^{X_2}a_2x_1}\Big)\mathrm{d}\mathcal{X} \\
&\quad + f(-a_1i_k)f(\xi i_k)\int_{\pi^{k+Z_1-(X_2-X_1)+1}\mathfrak{O}_\nu}\mathrm{d}x_1.
\end{aligned}$$

Again we may disregard the first term but now it is the second and fourth integrals which are easily solved. Therefore in order to calculate $I_1^{(\xi)}(i_k)$ we must concentrate on the term,

$$\tilde{I}_1^{(\xi)}(i_k) := f(\xi i_k)\int_{\pi^{k+Z_1-(X_2-X_1)}\mathfrak{O}_\nu\backslash\pi^{k+Z_1-(X_2-X_1)+1}\mathfrak{O}_\nu} f\Big(\frac{\pi^{X_2-Z_1}a_2c_1x_1-\pi^{X_1}a_1(\pi^{Z_2}x_2+\ldots+\pi^k i_k)}{\pi^{X_2}a_2x_1}\Big)\mathrm{d}\mathcal{X}.$$

Once again substituting,

$$x_1 \longmapsto \pi^{Z_1-(X_2-X_1)+k}x_1, \qquad \mathrm{d}x_1 \longmapsto \mathrm{d}x_1 \pmod m,$$

this integral becomes,

$$\tilde{I}_1^{(\xi)}(i_k) = f(\xi i_k) \int_{\mathfrak{O}_\nu} \int_{\substack{x_1 \in \mathfrak{O}_\nu^\times \\ c_1a_2x_1 - a_1i_k \not\equiv 0(\pi)}} f(c_1a_2x_1 - a_1i_k) \mathrm{d}\mathcal{X}$$
$$+ f(\xi i_k) \int_{\mathfrak{O}_\nu} \int_{\substack{x_1 \in \mathfrak{O}_\nu^\times \\ c_1a_2x_1 - a_1i_k \equiv 0(\pi)}} f\Big(\tfrac{(c_1a_2x_1 - a_1i_k) - \pi a_1 i_{k+1} - \ldots - \pi^{Z_2 - k} a_1 x_2}{\pi^{Z_1} a_2 x_1}\Big) \mathrm{d}\mathcal{X}.$$

However, in this last integral we have $c_1a_2x_1 - a_1i_k \equiv 0 \pmod{\pi}$. This implies that,

$$f(\pi^{Z_1} a_2 x_1) = f(\pi^{Z_1} a_2 (a_1 c_1^{-1} a_2^{-1} i_k)).$$

So in fact we find,

$$\tilde{I}_1^{(\xi)}(i_k) = f(\xi i_k) \int_{\mathfrak{O}_\nu} \int_{\substack{x_1 \in \mathfrak{O}_\nu^\times \\ c_1a_2x_1 - a_1i_k \not\equiv 0(\pi)}} f(c_1a_2x_1 - a_1i_k) \mathrm{d}\mathcal{X}$$
$$+ f(\xi i_k) \int_{\mathfrak{O}_\nu} \int_{\substack{x_1 \in \mathfrak{O}_\nu^\times \\ c_1a_2x_1 - a_1i_k \equiv 0(\pi)}} f\Big(\tfrac{(c_1a_2x_1 - a_1i_k) - \pi a_1 i_{k+1} - \ldots - \pi^{Z_2 - k} a_1 x_2}{\pi^{Z_1} a_2 (a_1 c_1^{-1} a_2^{-1}) i_k}\Big) \mathrm{d}\mathcal{X}$$
$$= K_2^{(\xi)}(i_k) + K_3^{(\xi)}(i_k),$$

with these integrals identical to those calculated in Lemmas 5.5.2 and 5.5.3. Using these results we find,

$$\sum_{i_k \in \Pi \setminus 0} \tilde{I}^{(\xi)}(i_k) = \sum_{i_k \in \Pi \setminus 0} K_2^{(\xi)}(i_k) + \sum_{i_k \in \Pi \setminus 0} K_3^{(\xi)}(i_k)$$
$$= \Big( \frac{(\rho - 1)}{m} - \sum_{i_k \in \Pi \setminus 0} f(-a_1 i_k) f(\xi i_k) \Big)$$
$$+ \Big( \frac{(\rho - 1)}{m} Z_1 + \sum_{i_k \in \Pi \setminus 0} f(a_2(a_1 c_1^{-1} a_2^{-1}) i_k) f(\xi i_k) \Big).$$

In order to complete this section we must return to the calculation of the integrals $I_1^{(\xi)}(i_k)$. Using our previous results we are able to conclude that whenever $(X_2 - X_1) - Z_1 < 0 \leq k$ these integrals satisfy,

$$\sum_{i_k \in \Pi \setminus 0} I_1^{(\xi)}(i_k) = \frac{(\rho - 1)}{m}(Z_1 - (X_2 - X_1) - 1) + \frac{(\rho - 1)}{m}(Z_1 + 1) - \sum_{i_k \in \Pi \setminus 0} f(-a_1 i_k) f(\xi i_k)$$
$$+ \sum_{i_k \in \Pi \setminus 0} f(a_2(a_1 c_1^{-1} a_2^{-1}) i_k) f(\xi i_k) + \sum_{i_k \in \Pi \setminus 0} f(-a_1 i_k) f(\xi i_k)$$
$$= \frac{(\rho - 1)}{m}((X_2 - X_1)) + \sum_{i_k \in \Pi \setminus 0} f((a_1 c_1^{-1}) i_k) f(\xi i_k).$$

Having completed our calculations concerning the integrals $I_1^{(\xi)}(i_k)$ we must now turn our attention towards solving the integrals $I_2^{(\xi)}(i_k)$.

### The Integrals $I_2^{(\xi)}(i_k)$

Having employed the usual substitutions for $I_2^{(\xi)}(i_k)$ we find that these integrals are dependent only on $i_k$ and are given by,

$$\begin{aligned}I_2^{(\xi)}(i_k) &= \int_{\mathfrak{D}_v^2} f\left(\frac{\pi^{X_2-Z_1}a_2c_1(x_1+\pi^{-Z_2}c_2(\pi^{Z_2}x_2+\ldots+\pi^k i_k))-\pi^{X_1}a_1(\pi^{Z_2}x_2+\ldots+\pi^k i_k)}{\pi^{X_2}a_2x_1+\pi^{X_2-Z_2}a_2c_2(\pi^{Z_2}x_2+\ldots+\pi^k i_k)}\right)f(\xi c_2 i_k)\mathrm{d}\mathcal{X} \\ &= \int_{\mathfrak{D}_v^2} f\left(\frac{\pi^{X_2-Z_1}a_2c_1x_1+\pi^{X_2-Z_1-Z_2}a_2c_1c_2(\ldots+\pi^k i_k))-\pi^{X_1}a_1(+\ldots+\pi^k i_k)}{\pi^{X_2}a_2x_1+\pi^{X_2-Z_2}a_2c_2(\pi^{Z_2}x_2+\ldots+\pi^k i_k)}\right)f(\xi c_2 i_k)\mathrm{d}\mathcal{X}.\end{aligned}$$

Before we begin let us note that we already have,

$$X_2-Z_1 > X_2-Z_1-(Z_2-k) \quad \text{and} \quad X_2 > X_2-(Z_2-k) > X_2-Z_1-(Z_2-k).$$

Therefore whenever $X_2-X_1 \neq Z_1+Z_2$ we find,

$$f\left(\frac{\pi^{X_2-Z_1}a_2c_1x_1+\pi^{X_2-Z_1-Z_2}a_2c_1c_2(\ldots+\pi^k i_k))-\pi^{X_1}a_1(\ldots+\pi^k i_k)}{\pi^{X_2}a_2x_1+\pi^{X_2-Z_2}a_2c_2(\pi^{Z_2}x_2+\ldots+\pi^k i_k)}\right) = f(\pi^{X_2-Z_1-Z_2+k}a_2c_1c_2i_k - \pi^{X_1+k}a_1i_k).$$

Using this fact we see that we must begin by splitting the integrals $I_2^{(\xi)}(i_k)$ into three distinct cases. We shall start by considering the two simplest of these.

**Case (1)** : $X_2-X_1 < Z_1+Z_2$

Since $X_2-X_1 \neq Z_1+Z_2$ we simply find,

$$\sum_{i_k\in\Pi\backslash 0} I_2^{(\xi)}(i_k) = \sum_{i_k\in\Pi\backslash 0} f(c_1c_2a_2i_k)f(\xi c_2 i_k).$$

**Case (2)** : $X_2-X_1 > Z_1+Z_2$

As with the previous case, since $X_2-X_1 \neq Z_1+Z_2$, we have

$$\sum_{i_k\in\Pi\backslash 0} I_2^{(\xi)}(i_k) = \sum_{i_k\in\Pi\backslash 0} f(-a_1i_k)f(\xi c_2 i_k).$$

We must now consider the last of the three cases when $X_2-X_1 = Z_1+Z_2$. Unfortunately this is not so simple.

**Case (3)** : $X_2-X_1 = Z_1+Z_2$

For this case we are able to write,

$$I_2^{(\xi)}(i_k) = f(\xi c_2 i_k)\int_{\mathfrak{D}_v^2} f\left(\frac{\pi^{X_2-Z_1}a_2c_1x_1+\pi^{X_2-Z_1-Z_2}(a_2c_1c_2-a_1)(\ldots+\pi^k i_k)}{\pi^{X_2-Z_2+k}a_2c_2i_k}\right)\mathrm{d}\mathcal{X}. \tag{5.3}$$

In order to calculate this integral we shall need to split this up into many more cases.

**Case (3.1)** : $(a_2c_1c_2 - a_1) = 0$

First of all we shall consider the case when $(a_2c_1c_2 - a_1) = 0$. Here we find that the integral $I_2^{(\xi)}(i_k)$ satisfies,

$$\begin{aligned} I_2^{(\xi)}(i_k) &= f(\xi c_2 i_k)\int_{\mathfrak{D}_\nu^2} f\left(\tfrac{\pi^{X_2-Z_1}a_2c_1x_1}{\pi^{X_2-Z_2+k}a_2c_2i_k}\right)\mathrm{d}\mathcal{X} \\ &= \begin{cases} f(a_2c_2i_k)f(\xi c_2 i_k) & Z_2 - Z_1 > k \\ f(a_2c_2i_k)f(\xi c_2 i_k) + f(\xi c_2 i_k)(\rho^{k-(Z_2-Z_1)+1} - 1)/m & Z_2 - Z_1 \le k. \end{cases} \end{aligned}$$

Therefore we shall have two possible results:

**(3.1.1)** $(Z_2 - Z_1) > k$ :

$$\sum_{i_k \in \Pi \backslash 0} I_2^{(\xi)}(i_k) = \sum_{i_k \in \Pi \backslash 0} f(a_2c_2i_k)f(\xi c_2 i_k),$$

**(3.1.2)** $(Z_2 - Z_1) \le k$ :

$$\sum_{i_k \in \Pi \backslash 0} I_2^{(\xi)}(i_k) = \sum_{i_k \in \Pi \backslash 0} f(a_2c_2i_k)f(\xi c_2 i_k) + \frac{(\rho - 1)}{m}(k - (Z_2 - Z_1) + 1).$$

In the remaining cases we shall no longer have $(a_2c_1c_2 - a_1) = 0$. So, as in the statement of the proof, we must define

$$(a_2c_1c_2 - a_1) := \pi^H h \quad \text{for some } H \ge 0 \text{ and } |h|_\nu = 1.$$

Looking back to equation (5.3), when $(a_2c_1c_2 - a_1) \neq 0$, we shall be required to consider two distinct cases $X_2 - Z_2 + k < X_2 - Z_1$ and $X_2 - Z_2 + k \ge X_2 - Z_1$.

**Case (3.2)** : $Z_2 - Z_1 > k$

If we return to equation (5.3) we see that in this case we find,

$$\begin{aligned} I_2^{(\xi)}(i_k) &= f(\xi c_2 i_k)\int_{\mathfrak{D}_\nu^2} f\left(\tfrac{\pi^{X_2-Z_1-Z_2}(a_2c_1c_2-a_1)(\pi^{Z_2}x_2+\ldots+\pi^k i_k)}{\pi^{X_2-Z_2+k}a_2c_2i_k}\right)\mathrm{d}\mathcal{X} \\ &= f(\xi c_2 i_k)\int_{\mathfrak{D}_\nu^2} f\left(\tfrac{\pi^{X_2-Z_1-Z_2+k+H}h\, i_k}{\pi^{X_2-Z_2+k}a_2c_2i_k}\right)\mathrm{d}\mathcal{X} \\ &= f(\xi c_2 i_k)\int_{\mathfrak{D}_\nu^2} f\left(\tfrac{\pi^{H-Z_1}h\, i_k}{a_2c_2i_k}\right)\mathrm{d}\mathcal{X} \\ &= \begin{cases} f(hi_k)f(\xi c_2 i_k) & H \le Z_1 \\ f(a_2c_2i_k)f(\xi c_2 i_k) & H > Z_1 \end{cases} \end{aligned}$$

In conclusion, whenever we have $Z_2 - Z_1 > k$ we shall have two distinct results:

**(3.2.1)** $H \leq Z_1 \;\Rightarrow\; (a_2c_1c_2 - a_1) \not\equiv 0 \ (\pi^{Z_1+1})$ :

$$\sum_{i_k \in \Pi \backslash 0} I_2^{(\xi)}(i_k) = \sum_{i_k \in \Pi \backslash 0} f(h\ i_k) f(\xi c_2 i_k),$$

**(3.2.2)** $H > Z_1 \;\Rightarrow\; (a_2c_1c_2 - a_1) \equiv 0 \ (\pi^{Z_1+1})$ :

$$\sum_{i_k \in \Pi \backslash 0} I_2^{(\xi)}(i_k) = \sum_{i_k \in \Pi \backslash 0} f(a_2c_2i_k) f(\xi c_2 i_k).$$

**Case (3.3)** : $Z_2 - Z_1 \leq k$

In this case equation (5.3) becomes,

$$I_2^{(\xi)}(i_k) = f(\xi c_2 i_k) \int_{\mathfrak{O}_\nu^2} f\left(\substack{\pi^{X_2-Z_1}a_2c_1x_1+\pi^{X_2-Z_1-Z_2+k+H}(hi_k+O(\pi)) \\ \pi^{X_2-Z_2+k}a_2c_2i_k}\right) \mathrm{d}\mathcal{X}. \tag{5.4}$$

After some consideration we see that this will provide us with a further three possible cases to consider.

**Case (3.3.1)** : $H < Z_2 - k$

Suppose that $H < Z_2 - k$ then we simply find,

$$I_2^{(\xi)}(i_k) = \int_{\mathfrak{O}_\nu^2} f(hi_k) f(\xi c_2 i_k) \mathrm{d}\mathcal{X} = f(h\ i_k) f(\xi c_2 i_k).$$

Thus, whenever $(a_2c_1c_2 - a_1) \not\equiv 0 \ (\pi^{Z_2-k})$ we have,

$$\sum_{i_k \in \Pi \backslash 0} I_2^{(\xi)}(i_k) = \sum_{i_k \in \Pi \backslash 0} f(hi_k) f(\xi c_2 i_k).$$

**Case (3.3.2)** : $Z_2 - k \leq H \leq Z_1$

In this case we must calculate,

$$\begin{aligned}
I_2^{(\xi)}(i_k) &= f(\xi c_2 i_k) \int_{\mathfrak{O}_\nu^2} f\left(\substack{\pi^{X_2-Z_1}a_2c_1x_1+\pi^{X_2-Z_1-Z_2+k+H}h(i_k+\pi i_{k+1}+\ldots+\pi^{Z_2-k}x_2) \\ \pi^{X_2-Z_2+k}a_2c_2i_k}\right) \mathrm{d}\mathcal{X} \\
&= f(\xi c_2 i_k) \int_{\mathfrak{O}_\nu \backslash \pi^{H-(Z_2-k)}\mathfrak{O}_\nu} f(a_2c_1x_1) \mathrm{d}x_1 \\
&\quad + f(\xi c_2 i_k) \int_{\mathfrak{O}_\nu} \int_{\pi^{H-(Z_2-k)}\mathfrak{O}_\nu \backslash \pi^{H-(Z_2-k)+1}\mathfrak{O}_\nu} f\left(\substack{\pi^{X_2-Z_1}a_2c_1x_1+\pi^{X_2-Z_1-Z_2+k+H}h(i_k+\ldots) \\ \pi^{X_2-Z_2+k}a_2c_2i_k}\right) \mathrm{d}\mathcal{X} \\
&\quad + f(hi_k) f(\xi c_2 i_k) \int_{\mathfrak{O}_\nu} \int_{\pi^{H-(Z_2-k)+1}\mathfrak{O}_\nu} \mathrm{d}\mathcal{X}.
\end{aligned}$$

The only difficulty in solving this expression lies in calculating the integral,

$$\tilde{I}_2^{(\xi)}(i_k) = f(\xi c_2 i_k)\int_{\mathfrak{O}_\nu}\int_{\pi^{H-(Z_2-k)}\mathfrak{O}_\nu\setminus\pi^{H-(Z_2-k)+1}\mathfrak{O}_\nu} f\Big(\tfrac{\pi^{X_2-Z_1}a_2c_1x_1+\pi^{X_2-Z_1-Z_2+k+H}(hi_k+O(\pi))}{\pi^{X_2-Z_2+k}a_2c_2i_k}\Big)\mathrm{d}\mathcal{X}.$$

Using the substitution $x_1 \longmapsto \pi^{H-(Z_2-k)}x_1$ this becomes,

$$\begin{aligned}
\tilde{I}_2^{(\xi)}(i_k) &= f(\xi c_2 i_k)\int_{\mathfrak{O}_\nu}\int_{\substack{x_1\in\mathfrak{O}_\nu^\times\\ a_2c_1x_1+hi_k\not\equiv 0(\pi)}} f(a_2c_1x_1+hi_k)\mathrm{d}\mathcal{X}\\
&\quad + f(\xi c_2 i_k)\int_{\mathfrak{O}_\nu}\int_{\substack{x_1\in\mathfrak{O}_\nu^\times\\ a_2c_1x_1+hi_k\equiv 0(\pi)}} f\Big(\tfrac{(a_2c_1x_1+hi_k)+\pi hi_{k+1}+\dots}{\pi^{Z_1-H}a_2c_2i_k}\Big)\mathrm{d}\mathcal{X}\\
&= K_2^{(\xi)}(i_k)+K_3^{(\xi)}(i_k).
\end{aligned}$$

Finally, putting everything back together and using the results given in Lemmas 5.5.2 and 5.5.3, we may conclude that whenever we have

$$(a_2c_1c_2-a_1)\equiv 0(\pi^{Z_2-k}) \quad\text{and}\quad (a_2c_1c_2-a_1)\not\equiv 0(\pi^{Z_1+1}),$$

the integrals $I_2^{(\xi)}(i_k)$ satisfy,

$$\begin{aligned}
\sum_{i_k\in\Pi\setminus 0} I_2^{(\xi)}(i_k) &= \frac{(\rho-1)}{m}(H-(Z_2-k))+\frac{(\rho-1)}{m}-\sum_{i_k\in\Pi\setminus 0} f(hi_k)f(\xi c_2 i_k)\\
&\quad +\frac{(\rho-1)}{m}(Z_1-H)+\sum_{i_k\in\Pi\setminus 0} f(a_2c_2i_k)f(\xi c_2i_k)+\sum_{i_k\in\Pi\setminus 0} f(hi_k)f(\xi c_2i_k)\\
&= \frac{(\rho-1)}{m}((Z_1-Z_2)+(k+1))+\sum_{i_k\in\Pi\setminus 0} f(a_2c_2i_k)f(\xi c_2i_k).
\end{aligned}$$

**Case (3.3.3) : $Z_1 < H$**

For the final case when $H > Z_1$ we simply find that,

$$\begin{aligned}
I_2^{(\xi)}(i_k) &= f(\xi c_2 i_k)\int_{\mathfrak{O}_\nu^2} f\Big(\tfrac{\pi^{X_2-Z_1}a_2c_1x_1}{\pi^{X_2-Z_2+k}a_2c_2i_k}\Big)\mathrm{d}\mathcal{X}\\
&= f(\xi c_2 i_k)\int_{\mathfrak{O}_\nu\setminus\pi^{k-(Z_2-Z_1)+1}\mathfrak{O}_\nu} f(a_2c_1x_1)\mathrm{d}x_1 \;+\; f(a_2c_2i_k)f(\xi c_2i_k)\int_{\pi^{k-(Z_2-Z_1)+1}\mathfrak{O}_\nu}\mathrm{d}x_1\\
&= \frac{1}{m}(\rho^{k-(Z_2-Z_1)+1}-1)f(\xi c_2i_k) \;+\; f(a_2c_2i_k)f(\xi c_2i_k).
\end{aligned}$$

Therefore, whenever $(a_2c_1c_2-a_1)\equiv 0(\pi^{Z_1+1})$ we simply find that,

$$\sum_{i_k\in\Pi\setminus 0} I_2^{(\xi)}(i_k) = \frac{(\rho-1)}{m}(k-(Z_2-Z_1)+1) \;+\; \sum_{i_k\in\Pi\setminus 0} f(a_2c_2i_k)f(\xi c_2i_k).$$

Having completed all of the calculations concerning the integrals $I_1^{(\xi)}(i_k)$ and $I_2^{(\xi)}(i_k)$, we are now able to consider each of the possible cases which arise when evaluating the cocycle $dec_\nu$. Since we have seen these calculations many times throughout this chapter we shall leave some of the detail to the reader.

**Results for** $(X_2 - X_1) - Z_1 < 0$ **:**

| | $0 \le k < Z_2$ |
|---|---|
| $I_1^{(\xi)}(i_k)$ | Case (iii) |
| $I_2^{(\xi)}(i_k)$ | Case (1) |

$$\begin{aligned} dec_\nu(n_1\alpha_2\eta_\psi, n_2) &= \prod_{k=0}^{Z_2-1}\prod_{\xi\in\mu_m} \xi^{\sum_{i_k} I^{(\xi)}(i_k)} \prod_{k=0}^{Z_2-1}\prod_{\xi\in\mu_m} \xi^{-\sum_{i_k} I_2^{(\xi)}(i_k)} \\ &= (-1)^{\frac{(\rho-1)}{m}Z_2(X_2-X_1)}\,(a_1/c_1,\pi)_{\nu,m}^{Z_2}\cdot(a_2c_1,\pi)_{\nu,m}^{-Z_2} \\ &= (a_1/a_2,\pi)_{\nu,m}^{Z_2}(c_1,\pi)_{\nu,m}^{-2Z_2}\quad(-1)^{\frac{(\rho-1)}{2^r}Z_2(X_2-X_1)}. \end{aligned}$$

**Results for** $0 \le (X_2 - X_1) - Z_1 < Z_2$ **:**

| | $0 \le k < (X_2 - X_1) - Z_1$ | $(X_2 - X_1) - Z_1 \le k < Z_2$ |
|---|---|---|
| $I_1^{(\xi)}(i_k)$ | Case (i) | Case (ii) |
| $I_2^{(\xi)}(i_k)$ | Case (1) | |

We must once again split the product of the integrals $I_1^{(\xi)}(i_k)$ into the two possible results depending on the value of $k$. This allows us to calculate,

$$\begin{aligned} dec_\nu(n_1\alpha_2\eta_\psi, n_2) &= \prod_{k=0}^{(X_2-X_1)-Z_1-1}\cdot\prod_{k=(X_2-X_1)-Z_1}^{Z_2-1}\left(\prod_{\xi\in\mu_m}\xi^{\sum_{i_k} I^{(\xi)}(i_k)}\right)\prod_{k=0}^{Z_2-1}\left(\prod_{\xi\in\mu_m}\xi^{-\sum_{i_k} I_2^{(\xi)}(i_k)}\right) \\ &= (-1)^{\frac{(\rho-1)}{m}\frac{((X_2-X_1)-Z_1)((X_2-X_1)-Z_1+1)}{2}}(-a_1,\pi)_{\nu,m}^{(X_2-X_1)-Z_1} \\ &\quad .(-1)^{\frac{(\rho-1)}{m}(X_2-X_1)(Z_2-((X_2-X_1)-Z_1))}\,(c_1a_2,\pi)_{\nu,m}^{-(Z_2-((X_2-X_1)-Z_1))} \\ &\quad (a_2,\pi)_{\nu,m}^{Z_2-((X_2-X_1)-Z_1)}(a_1,\pi)_{\nu,m}^{Z_2-((X_2-X_1)-Z_1)}.(c_1a_2,\pi)_{\nu,m}^{-Z_2} \\ &= (a_1/a_2,\pi)_{\nu,m}^{Z_2}\quad(c_1,\pi)_{\nu,m}^{(X_2-X_1)-Z_1}\quad(c_1,\pi)_{\nu,m}^{-2Z_2} \\ &\quad .\,(-1)^{\frac{(\rho-1)}{2^r}Z_2(X_2-X_1)}\,(-1)^{\frac{(\rho-1)}{2^r}\frac{(X_2-X_1)((X_2-X_1)+1)}{2}}\,(-1)^{\frac{(\rho-1)}{2^r}\frac{Z_1(Z_1+1)}{2}}. \end{aligned}$$

**Results for** $Z_2 < (X_2 - X_1) - Z_1$ :

| | $0 \le k < Z_2$ |
|---|---|
| $I_1^{(\xi)}(i_k)$ | Case (i) |
| $I_2^{(\xi)}(i_k)$ | Case (2) |

$$\begin{aligned} dec_\nu(n_1\alpha_2\eta_\psi, n_2) &= \prod_{k=0}^{Z_2-1}\prod_{\xi\in\mu_m} \xi^{\sum_{i_k} I^{(\xi)}(i_k)} \prod_{k=0}^{Z_2-1}\prod_{\xi\in\mu_m} \xi^{-\sum_{i_k} I_2^{(\xi)}(i_k)} \\ &= (-1)^{\frac{(\rho-1)}{m}\frac{Z_2(Z_2+1)}{2}} (a_1,\pi)_{\nu,m}^{Z_2} \cdot (a_1/c_2,\pi)_{\nu,m}^{-Z_2} \\ &= (c_2,\pi)_{\nu,m}^{Z_2} \quad (-1)^{\frac{(\rho-1)}{2^r}\frac{Z_2(Z_2+1)}{2}}. \end{aligned}$$

**Results for** $X_2 - X_1 = Z_1 + Z_2, \quad (a_2c_1c_2 - a_1) = 0$ :

We must now consider the case when $X_2 - X_1 = Z_1 + Z_2 \le Z_2$ but $(a_2c_1c_2 - a_1) = 0$. This gives us three more distinct cases to consider.

Let us note that, since $k < Z_2 = (X_2 - X_1) - Z_1$, for the integrals $I_1^{(\xi)}(i_k)$ we need only consider our result in Case (i).

- $Z_2 - Z_1 \le 0$ :

| | $0 \le k < Z_2$ |
|---|---|
| $I_1^{(\xi)}(i_k)$ | Case (i) |
| $I_2^{(\xi)}(i_k)$ | Case (3.1.2) |

$$\begin{aligned} dec_\nu(n_1\alpha_2\eta_\psi, n_2) &= (-1)^{\frac{(\rho-1)}{m}\frac{Z_2(Z_2+1)}{2}} (-a_1,\pi)_{\nu,m}^{Z_2} \cdot (-1)^{\frac{(\rho-1)}{m}\frac{Z_2(Z_2+1)}{2}} (-1)^{\frac{(\rho-1)}{m}Z_2(Z_2-Z_1)} (a_2,\pi)_{\nu,m}^{-Z_2} \\ &= (a_1/a_2,\pi)_{\nu,m}^{Z_2} \quad (-1)^{\frac{(\rho-1)}{2^r}Z_1Z_2}. \end{aligned}$$

- $0 < Z_2 - Z_1 < Z_2$ :

| | $0 \le k < (X_2 - X_1)$ | $(X_2 - X_1) \le k < Z_2$ |
|---|---|---|
| $I_1^{(\xi)}(i_k)$ | Case (i) | |
| $I_2^{(\xi)}(i_k)$ | Case (3.1.1) | Case (3.1.2) |

$$\begin{aligned} dec_\nu(n_1\alpha_2\eta_\psi, n_2) &= (-1)^{\frac{(\rho-1)}{m}\frac{Z_2(Z_2+1)}{2}} (-a_1,\pi)_{\nu,m}^{Z_2} \cdot (-1)^{\frac{(\rho-1)}{m}\frac{Z_1(Z_1+1)}{2}} (a_2,\pi)_{\nu,m}^{-Z_2} \\ &= (a_1/a_2,\pi)_{\nu,m}^{Z_2} \quad (-1)^{\frac{(\rho-1)}{2^r}\frac{Z_2(Z_2+1)}{2}} (-1)^{\frac{(\rho-1)}{2^r}Z_2} (-1)^{\frac{(\rho-1)}{2^r}\frac{Z_1(Z_1+1)}{2}}. \end{aligned}$$

- $0 < Z_2 - Z_1 = Z_2, \ Z_1 = 0:$

| | $0 \le k < Z_2$ |
|---|---|
| $I_1^{(\xi)}(i_k)$ | Case (i) |
| $I_2^{(\xi)}(i_k)$ | Case (3.1.1) |

$$dec_\nu(n_1\alpha_2\eta_\psi, n_2) = (-1)^{\frac{(\rho-1)}{m}\frac{Z_2(Z_2+1)}{2}}(-a_1,\pi)_{\nu,m}^{Z_2} \cdot (a_2,\pi)_{\nu,m}^{-Z_2}$$
$$= (a_1/a_2,\pi)_{\nu,m}^{Z_2} \quad (-1)^{\frac{(\rho-1)}{2^r}\frac{Z_2(Z_2+1)}{2}}\,(-1)^{\frac{(\rho-1)}{2^r}Z_2}.$$

**Results for** $X_2 - X_1 = Z_1 + Z_2, \quad (a_2c_1c_2 - a_1) \neq 0:$

Once again, since $k < Z_2 = (X_2 - X_1) - Z_1$, for the integral $I_1^{(\xi)}(i_k)$ we shall only need to consider Case (i).

Let us remind ourselves that whenever $(a_2c_1c_2 - a_1) \neq 0$ we have defined,

$$(a_2c_1c_2 - a_1) := \pi^H h \quad \text{for some } H \ge 0 \quad \text{and where } \mid h \mid_\nu = 1.$$

As with the previous section we shall begin by considering the three distinct cases depending on the value of $Z_2 - Z_1$. Then, in each of these cases, we shall have to further consider the value of the exponent $H$.

- $Z_2 - Z_1 \le 0:$

In this case we shall have $Z_2 - Z_1 \le k$ for all $0 \le k < Z_2$. Therefore, depending on the value of the integer $H \ge 0$, results for the integrals $I_2^{(\xi)}(i_k)$ are given in Case (3.3).

- ◦ $H = 0:$

| | $0 \le k < Z_2$ |
|---|---|
| $I_1^{(\xi)}(i_k)$ | Case (i) |
| $I_2^{(\xi)}(i_k)$ | Case (3.3.1) |

$$dec_\nu(n_1\alpha_2\eta_\psi, n_2) = (-1)^{\frac{(\rho-1)}{m}\frac{Z_2(Z_2+1)}{2}}(-a_1,\pi)_{\nu,m}^{Z_2} \cdot (h/c_2,\pi)_{\nu,m}^{-Z_2}$$
$$= (a_1c_2/h,\pi)_{\nu,m}^{Z_2} \quad (-1)^{\frac{(\rho-1)}{2^r}\frac{Z_2(Z_2+1)}{2}}\,(-1)^{\frac{(\rho-1)}{2^r}Z_2}.$$

∘ $1 \leq H < Z_2$ :

| | $0 \leq k < Z_2 - H$ | $Z_2 - H \leq k < Z_2$ |
|---|---|---|
| $I_1^{(\xi)}(i_k)$ | Case (i) | |
| $I_2^{(\xi)}(i_k)$ | Case (3.3.1) | Case (3.3.2) |

$$\begin{aligned} dec_\nu(n_1\alpha_2\eta_\psi, n_2) &= (-1)^{\frac{(\rho-1)}{m}\frac{Z_2(Z_2+1)}{2}}(-a_1,\pi)_{\nu,m}^{Z_2} \cdot (h/c_2,\pi)_{\nu,m}^{H-Z_2} \cdot \\ &\quad (-1)^{\frac{(\rho-1)}{m}(Z_1-H)H}(-1)^{\frac{(\rho-1)}{m}\frac{H(H+1)}{2}}(a_2,\pi)_{\nu,m}^{-H} \\ &= (a_1c_2/h,\pi)_{\nu,m}^{Z_2} \quad (h/a_2c_2,\pi)_{\nu,m}^{H} \\ &\quad (-1)^{\frac{(\rho-1)}{2^r}\frac{Z_2(Z_2+1)}{2}}(-1)^{\frac{(\rho-1)}{2^r}Z_2}(-1)^{\frac{(\rho-1)}{2^r}\frac{H(H-1)}{2}}(-1)^{\frac{(\rho-1)}{2^r}Z_1H}. \end{aligned}$$

∘ $Z_2 \leq H \leq Z_1$ :

| | $0 \leq k < Z_2$ |
|---|---|
| $I_1^{(\xi)}(i_k)$ | Case (i) |
| $I_2^{(\xi)}(i_k)$ | Case (3.3.2) |

$$\begin{aligned} dec_\nu(n_1\alpha_2\eta_\psi, n_2) &= (-1)^{\frac{(\rho-1)}{m}\frac{Z_2(Z_2+1)}{2}}(-a_1,\pi)_{\nu,m}^{Z_2} \cdot (-1)^{\frac{(\rho-1)}{m}(Z_1-Z_2)Z_2}(-1)^{\frac{(\rho-1)}{m}\frac{Z_2(Z_2+1)}{2}}(a_2,\pi)_{\nu,m}^{-Z_2} \\ &= (a_1/a_2,\pi)_{\nu,m}^{Z_2} \quad (-1)^{\frac{(\rho-1)}{2^r}Z_1Z_2}. \end{aligned}$$

∘ $Z_1 < H$ :

| | $0 \leq k < Z_2$ |
|---|---|
| $I_1^{(\xi)}(i_k)$ | Case (i) |
| $I_2^{(\xi)}(i_k)$ | Case (3.3.3) |

$$\begin{aligned} dec_\nu(n_1\alpha_2\eta_\psi, n_2) &= (-1)^{\frac{(\rho-1)}{m}\frac{Z_2(Z_2+1)}{2}}(-a_1,\pi)_{\nu,m}^{Z_2} \cdot (-1)^{\frac{(\rho-1)}{m}(Z_2-Z_1)Z_2}(-1)^{\frac{(\rho-1)}{m}\frac{Z_2(Z_2+1)}{2}}(a_2,\pi)_{\nu,m}^{-Z_2} \\ &= (a_1/a_2,\pi)_{\nu,m}^{Z_2} \quad (-1)^{\frac{(\rho-1)}{2^r}Z_1Z_2}. \end{aligned}$$

- $0 < Z_2 - Z_1 < Z_2$ :

For the integrals $I_2^{(\xi)}(i_k)$ we can no longer just consider the case $Z_2 - Z_1 \leq k$. That is, for this case we shall have

$$k < Z_2 - Z_1 \quad \text{for } 0 \leq k < Z_2 - Z_1 \qquad \text{and} \qquad Z_2 - Z_1 \leq k \quad \text{for } Z_2 - Z_1 \leq k < Z_2,$$

so we must consider the results of both Cases (3.2) and (3.3). Once again, for the results in Case (3.3) we must consider the four possible values of the integer $H \geq 0$.

- $H = 0$ :

| | $0 \leq k < Z_2 - Z_1$ | $Z_2 - Z_1 \leq k < Z_2$ |
|---|---|---|
| $I_1^{(\xi)}(i_k)$ | Case (i) | |
| $I_2^{(\xi)}(i_k)$ | Case (3.2.1) | Case (3.3.1) |

$$\begin{aligned} dec_\nu(n_1\alpha_2\eta_\psi, n_2) &= (-1)^{\frac{(\rho-1)}{m}\frac{Z_2(Z_2+1)}{2}}(-a_1,\pi)_{\nu,m}^{Z_2} \cdot (h/c_2,\pi)_{\nu,m}^{-(Z_2-Z_1)} \\ &\qquad (h/c_2,\pi)_{\nu,m}^{-(Z_2-(Z_2-Z_1))} \\ &= (a_1c_2/h,\pi)_{\nu,m}^{Z_2} \quad (-1)^{\frac{(\rho-1)}{2^r}\frac{Z_2(Z_2+1)}{2}} (-1)^{\frac{(\rho-1)}{2^r}Z_2}. \end{aligned}$$

- $1 \leq H < Z_1$ :

| | $0 \leq k < Z_2 - Z_1$ | $Z_2 - Z_1 \leq k < Z_2 - H$ | $Z_2 - H \leq k < Z_2$ |
|---|---|---|---|
| $I_1^{(\xi)}(i_k)$ | Case (i) | | |
| $I_2^{(\xi)}(i_k)$ | Case (3.2.1) | Case (3.3.1) | Case (3.3.2) |

$$\begin{aligned} dec_\nu(n_1\alpha_2\eta_\psi, n_2) &= (-1)^{\frac{(\rho-1)}{m}\frac{Z_2(Z_2+1)}{2}}(-a_1,\pi)_{\nu,m}^{Z_2} \cdot (h/c_2,\pi)_{\nu,m}^{-(Z_2-Z_1)} \\ &\qquad (h/c_2,\pi)_{\nu,m}^{H-Z_2+(Z_2-Z_1)} \cdot (-1)^{\frac{(\rho-1)}{m}(Z_1-H)H}(-1)^{\frac{(\rho-1)}{m}\frac{H(H+1)}{2}}(a_2,\pi)_{\nu,m}^{-H} \\ &= (a_1c_2/h,\pi)_{\nu,m}^{Z_2} \quad (h/a_2c_2,\pi)_{\nu,m}^{H} \\ &\qquad (-1)^{\frac{(\rho-1)}{2^r}\frac{Z_2(Z_2+1)}{2}} (-1)^{\frac{(\rho-1)}{2^r}Z_2} (-1)^{\frac{(\rho-1)}{2^r}\frac{H(H-1)}{2}} (-1)^{\frac{(\rho-1)}{2^r}Z_1H}. \end{aligned}$$

∘ $H = Z_1$ :

| | $0 \le k < Z_2 - Z_1$ \| $Z_2 - Z_1 \le k < Z_2$ |
|---|---|
| $I_1^{(\xi)}(i_k)$ | Case (i) |
| $I_2^{(\xi)}(i_k)$ | Case (3.2.1) \| Case (3.3.2) |

$$\begin{aligned} dec_\nu(n_1\alpha_2\eta_\psi, n_2) &= (-1)^{\frac{(\rho-1)}{m}\frac{Z_2(Z_2+1)}{2}}(-a_1,\pi)_{\nu,m}^{Z_2} \cdot (h/c_2,\pi)_{\nu,m}^{-(Z_2-Z_1)} \\ &\quad (-1)^{\frac{(\rho-1)}{m}\frac{Z_1(Z_1+1)}{2}}(a_2,\pi)_{\nu,m}^{-Z_1} \\ &= (a_1c_2/h,\pi)_{\nu,m}^{Z_2} \quad (h/a_2c_2,\pi)_{\nu,m}^{Z_1} \\ &\quad (-1)^{\frac{(\rho-1)}{2^r}\frac{Z_2(Z_2+1)}{2}}(-1)^{\frac{(\rho-1)}{2^r}Z_2}(-1)^{\frac{(\rho-1)}{2^r}\frac{Z_1(Z_1+1)}{2}}. \end{aligned}$$

∘ $Z_1 < H$ :

| | $0 \le k < Z_2 - Z_1$ \| $Z_2 - Z_1 \le k < Z_2$ |
|---|---|
| $I_1^{(\xi)}(i_k)$ | Case (i) |
| $I_2^{(\xi)}(i_k)$ | Case (3.2.2) \| Case (3.3.3) |

$$\begin{aligned} dec_\nu(n_1\alpha_2\eta_\psi, n_2) &= (-1)^{\frac{(\rho-1)}{m}\frac{Z_2(Z_2+1)}{2}}(-a_1,\pi)_{\nu,m}^{Z_2} \cdot (a_2,\pi)_{\nu,m}^{-(Z_2-Z_1)} \\ &\quad \cdot (-1)^{\frac{(\rho-1)}{m}\frac{Z_1(Z_1+1)}{2}}(a_2/,\pi)_{\nu,m}^{(Z_2-Z_1)-Z_2} \\ &= (a_1/a_2,\pi)_{\nu,m}^{Z_2} \quad (-1)^{\frac{(\rho-1)}{2^r}\frac{Z_2(Z_2+1)}{2}}(-1)^{\frac{(\rho-1)}{2^r}Z_2}(-1)^{\frac{(\rho-1)}{2^r}\frac{Z_1(Z_1+1)}{2}}. \end{aligned}$$

- $0 < Z_2 - Z_1 = Z_2$ :

This is the simplest case since the above condition requires that $Z_1 = 0$. Therefore we always have $k < Z_2 - Z_1$ and we may restrict our attention to the results of Case (3.2) with $Z_1$ replaced by 0. Therefore this case will provide us with a further two possible results.

- $H = 0$ :

| | $0 \leq k < Z_2$ |
|---|---|
| $I_1^{(\xi)}(i_k)$ | Case (i) |
| $I_2^{(\xi)}(i_k)$ | Case (3.2.1) |

$$\begin{aligned} dec_\nu(n_1 \alpha_2 \eta_\psi, n_2) &= (-1)^{\frac{(\rho-1)}{m}\frac{Z_2(Z_2+1)}{2}} (-a_1, \pi)_{\nu,m}^{Z_2} \cdot (h/c_2, \pi)_{\nu,m}^{-Z_2} \\ &= (a_1 c_2/h, \pi)_{\nu,m}^{Z_2} \quad (-1)^{\frac{(\rho-1)}{2^r}\frac{Z_2(Z_2+1)}{2}} (-1)^{\frac{(\rho-1)}{2^r}Z_2}. \end{aligned}$$

- $H > 0$ :

| | $0 \leq k < Z_2$ |
|---|---|
| $I_1^{(\xi)}(i_k)$ | Case (i) |
| $I_2^{(\xi)}(i_k)$ | Case (3.2.2) |

$$\begin{aligned} dec_\nu(n_1 \alpha_2 \eta_\psi, n_2) &= (-1)^{\frac{(\rho-1)}{m}\frac{Z_2(Z_2+1)}{2}} (-a_1, \pi)_{\nu,m}^{Z_2} \cdot (a_2, \pi)_{\nu,m}^{-Z_2} \\ &= (a_1/a_2, \pi)_{\nu,m}^{Z_2} \quad (-1)^{\frac{(\rho-1)}{2^r}\frac{Z_2(Z_2+1)}{2}} (-1)^{\frac{(\rho-1)}{2^r}Z_2}. \end{aligned}$$

Finally, in order to prove the original statement of the theorem it simply remains for us to tidy up these results and simplify the various conditions. Having done so we eventually find the concise version of these results given on page 129. That is, by going through each result we see that it is indeed consistent with those given in the theorem.

□

To conclude this chapter we use the results given in the previous theorem to immediately deduce the following corollary.

**Corollary 5** *Let* $n_1 := n_{1,2}(\pi^{-Z_1}c_1)$, $n_2 := n_{1,2}(\pi^{-Z_2}c_2) \in N$ *with* $Z_2 > 0$ *and* $\eta_\psi \in \mathfrak{M}$ *be as previously defined. Then the cocycle* $dec_\nu$ *satisfies,*

- $Z_1 < 0$ :

$$dec_\nu\left(\begin{pmatrix} 1 & \pi^{-Z_1}c_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & \pi^{-Z_2}c_2 \\ 0 & 1 \end{pmatrix}\right) = dec_\nu(\eta_\psi, n_2) = 1.$$

- $0 \leq Z_1$ :

$$dec_\nu\left(\begin{pmatrix} 1 & \pi^{-Z_1}c_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & \pi^{-Z_2}c_2 \\ 0 & 1 \end{pmatrix}\right) = (c_1, \pi)_{\nu,m}^{-2Z_2}.$$

# Chapter 6

# Calculation of $dec_\nu$ on $N$

## 6.1 Introduction

Once again we begin by letting $k_\nu$ be a local field with valuation $\nu$ and a fixed uniformizing element $\pi$ in $\mathfrak{O}_\nu$ the ring of integers.

As we saw in the first chapter our choices of the sets $\Phi$ and $\Delta$ determine a Borel subgroup of $\mathrm{GL}_n(k_\nu)$ whose unipotent radical we denote by $N$. It is known that the group $N$ is generated by,

$$N = < n_{i,j}(x) : x \in k_\nu,\ (i,j) \in \Delta >,$$

where, for every $(i,j) \in \Phi^+$ and each $x \in k_\nu$, we define the matrix

$$n_{i,j}(x) = I_n + x.e_{i,j},$$

and $e_{i,j}$ is the elementary matrix with a 1 in the $(i,j)^{th}$ position and 0's elsewhere.

Throughout this chapter we shall be concerned with calculating the cocycle $dec_\nu$ on the space $N \times N$. Since we shall eventually find that Matsumoto's cocycle is trivial on this space, in the later chapters, we will in fact be able to show that the cocycle $dec_\nu$ splits on $N \times N$. Using the results found in this chapter we shall be able to go some way to finding that splitting.

Before we begin, for each $1 \leq i < j \leq n$, let us define the matrices

$$n_{i,j} := n_{i,j}(\pi^{-Z_{i,j}} c_{i,j}) \in N \quad \text{such that } |c_{i,j}|_\nu = 1.$$

## 6.2 The cocycle $dec_\nu$ on $N$

To be more precise, in this chapter we shall be considering the value of $dec_\nu(n, n_{i,j})$ where $n \in N$ is any matrix and $n_{i,j} \in N$ is as described above. In this section we shall consider two simple theorems which shall later help us with our calculations. We shall then discuss the general method we shall use to complete these calculations.

**Theorem 6.2.1** *Let $k_\nu$ be a local field with valuation $\nu$. For any matrices $g_1, g_2 \in \mathrm{GL}_n(k_\nu)$ and for each $n_{i,j} \in N$ with $Z_{i,j} < 0$ the cocycle $dec_\nu$ satisfies,*

$$dec_\nu(n_{i,j} g_1, g_2) = dec_\nu(g_1, g_2).$$

**PROOF OF THEOREM:**

Whenever $Z_{i,j} < 0$ the matrix $n_{i,j}$ satisfies,

$$n_{i,j} := n_{i,j}(\pi^{-Z_{i,j}} c_{i,j}) \equiv I_n \pmod{\pi}.$$

Therefore, by Lemma 1.4.4, we have $f n_{i,j} = f$ and our result follows immediately.

□

**Theorem 6.2.2** *Let $k_\nu$ be a local field with valuation $\nu$. For any matrix $g \in \mathrm{GL}_n(k_\nu)$ and for each $n_{i,j} \in N$ with $Z_{i,j} \leq 0$ the cocycle $dec_\nu$ satisfies,*

$$dec_\nu(g, n_{i,j}) = 1.$$

**PROOF OF THEOREM:**

Since we have $Z_{i,j} \leq 0$ we find that $n_{i,j} := n_{i,j}(\pi^{-Z_{i,j}} c_{i,j}) \in \mathrm{GL}_n(\mathfrak{O}_\nu)$. Using this fact our result follows by Theorem 1.4.1.

□

### 6.2.1 Our Method of Calculation for $dec_\nu(g, n_{i,j})$

Once again, it remains for us to find some way of calculating $dec_\nu(g, n_{i,j})$ when $Z_{i,j} > 0$. This is done by expanding on the ideas in the previous chapter.

Once again we define the set $\Pi$ to be a set of representatives for $\mathfrak{O}_\nu/\pi\mathfrak{O}_\nu$ including 0. We are then able to dissect the space $\mathfrak{O}_\nu^n$ into disjoint sets $A(i_o, \dots, i_{Z_{i,j}-1})$ defined such that,

$$\mathfrak{O}_\nu^n = \bigcup_{i_{Z_{i,j}-1}\in\Pi} \cdots \bigcup_{i_o\in\Pi} \left\{ \begin{pmatrix} x_1 \\ \vdots \\ \pi^{Z_{i,j}}x_j + \pi^{Z_{i,j}-1} i_{Z_{i,j}-1} + \dots + \pi i_1 + i_o \\ \vdots \\ x_n \end{pmatrix} : x_i \in \mathfrak{O}_\nu, \forall i \right\}$$

$$:= \bigcup_{i_{Z_{i,j}-1}\in\Pi} \cdots \bigcup_{i_o\in\Pi} A(i_o, \dots, i_{Z_{i,j}-1}).$$

If we now define,

$$I_1^{(\xi)}(i_o, \dots, i_{Z_{i,j}-1}) = \int_{A(i_o,\dots,i_{Z_{i,j}-1})} f(g\mathcal{X}) f(\xi\mathcal{X}) \mathrm{d}\mathcal{X}$$

$$\text{and} \quad I_2^{(\xi)}(i_o, \dots, i_{Z_{i,j}-1}) = \int_{n_{i,j}A(i_o,\dots,i_{Z_{i,j}-1})} f(g\mathcal{X}) f(\xi\mathcal{X}) \mathrm{d}\mathcal{X},$$

then, using this dissection we may write,

$$I(\xi) = \sum_{i_{Z_{i,j}-1}\in\Pi} \cdots \sum_{i_o\in\Pi} \left( I_1^{(\xi)}(i_o, \dots, i_{Z_{i,j}-1}) - I_2^{(\xi)}(i_o, \dots, i_{Z_{i,j}-1}) \right).$$

Since we once again find,

$$i_o = i_1 = \dots = i_{Z_{i,j}-1} = 0 \quad \Rightarrow \quad n_{i,j}A(i_o, \dots, i_{Z_{i,j}-1}) = A(i_o, \dots, i_{Z_{i,j}-1}),$$

we are again able to deduce that,

$$I_1^{(\xi)}(0, \dots, 0) - I_2^{(\xi)}(0, \dots, 0) = 0. \qquad (\dagger)$$

Finally, since throughout the course of this chapter we shall see that the integrals $I_1^{(\xi)}(0, ., 0, i_k, ., i_{Z_{i,j}-1})$ and $I_2^{(\xi)}(0, ., 0, i_k, ., i_{Z_{i,j}-1})$ depend only on $i_k$, we shall once again make the notation simpler by defining,

$$I_1^{(\xi)}(i_k) := I_1^{(\xi)}(0, ., 0, i_k, ., i_{Z_{i,j}-1}) \quad \text{and} \quad I_2^{(\xi)}(i_k) := I_2^{(\xi)}(0, ., 0, i_k, ., i_{Z_{i,j}-1}),$$

being careful to remember that we are in fact considering a set of similar integrals.

Using this new notation and (†) we find that the integral $I(\xi)$ satisfies,

$$I(\xi) = \sum_{k=0}^{Z_{i,j}-1}\Big(\sum_{i_{Z_{i,j}-1}\in\Pi}\cdots\sum_{i_{k+1}\in\Pi}\sum_{i_k\in\Pi\backslash 0} I_1^{(\xi)}(i_k)\Big) - \sum_{k=0}^{Z_{i,j}-1}\Big(\sum_{i_{Z_{i,j}-1}\in\Pi}\cdots\sum_{i_{k+1}\in\Pi}\sum_{i_k\in\Pi\backslash 0} I_2^{(\xi)}(i_k)\Big)$$
$$\equiv \sum_{k=0}^{Z_{i,j}-1}\sum_{i_k\in\Pi\backslash 0} I_1^{(\xi)}(i_k) - \sum_{k=0}^{Z_{i,j}-1}\sum_{i_k\in\Pi\backslash 0} I_2^{(\xi)}(i_k) \pmod{m}.$$

Therefore, throughout this chapter we shall again be looking for the integrals $I_1^{(\xi)}(i_k)$ and $I_2^{(\xi)}(i_k)$ with $i_k$ strictly non-zero. Then, using Lemma 2.0.5 and the fact that $f$ is fundamental, we will be able to use to reconstruct the expression,

$$dec_\nu(g, n_{i,j}(\pi^{-Z_{i,j}}c_{i,j})) = \prod_{\xi\in\mu_m} \xi^{\sum_{k=0}^{Z_{i,j}-1}\sum_{i_k\in\Pi\backslash 0} I_1^{(\xi)}(i_k)} \prod_{\xi\in\mu_m} \xi^{-\sum_{k=0}^{Z_{i,j}-1}\sum_{i_k\in\Pi\backslash 0} I_2^{(\xi)}(i_k)}.$$

As with the previous chapter, when considering the integrals $I_1^{(\xi)}(i_k)$ and $I_2^{(\xi)}(i_k)$ we shall continue to use Lemmas 5.2.2 and 5.2.3 to ignore terms independent of $i_k$ and only consider the terms independent of $\xi$ modulo 2. With respect to this we shall again abuse the notation with regard to the equals sign.

## 6.3 $dec_\nu$ on $N$ in $\mathrm{GL}_n(k_\nu)$

In this section we shall concentrate on calculating $dec_\nu(n_{i,j}, n_{i,j})$ where the matrix $n_{i,j} \in N \subset \mathrm{GL}_n(k_\nu)$ is as previously described. This important result will eventually allow us to describe the splitting of the cocycle $dec_\nu$ on each of the generators $n_{i,j}$ of $N$.

**Theorem 6.3.1** *Let $k_\nu$ be a local field with valuation $\nu$. Then for each,*

$$n_{i,j} := n_{i,j}(\pi^{-Z_{i,j}}c_{i,j}) \in N \qquad with \quad Z_{i,j} > 0,$$

*the cocycle $dec_\nu$ satisfies,*

$$dec_\nu(n_{i,j}, n_{i,j}) = (c_{i,j}/2, \pi)_{\nu,m}^{Z_{i,j}}(-1)^{\frac{(\rho-1)}{2^r}\frac{Z_{i,j}(Z_{i,j}+1)}{2}(j-1)}(-1)^{\frac{(\rho-1)}{2^r}\frac{Z_{i,j}(Z_{i,j}-1)}{2}(n-j)}.$$

**PROOF OF THEOREM:**

As we stated in the previous section we shall calculate the value of the cocycle by first calculating the integrals $I_1^{(\xi)}(i_k)$ and $I_2^{(\xi)}(i_k)$.

### The Integrals $I_1^{(\xi)}(i_k)$

Having made the substitutions,

$$x_j \longmapsto \pi^{Z_{i,j}}x_j + \pi^{Z_{i,j}-1}i_{Z_{i,j}-1} + \ldots + \pi^k i_k \qquad \Rightarrow \mathrm{d}x_j \longmapsto |\pi^{Z_{i,j}}|_\nu \mathrm{d}x_j \equiv \mathrm{d}x_j \pmod m,$$

we are able to write,

$$\begin{aligned} I_1^{(\xi)}(i_k) &= \int_{\mathfrak{O}_\nu^n} f\begin{pmatrix} x_1 \\ \vdots \\ x_i+\pi^{-Z_{i,j}}c_{i,j}(\pi^{Z_{i,j}}x_j+\ldots+\pi^k i_k) \\ \vdots \\ (\pi^{Z_{i,j}}x_j\ldots+\pi^k i_k) \\ \vdots \\ x_n \end{pmatrix} f\begin{pmatrix} \xi x_1 \\ \vdots \\ \xi x_i \\ \vdots \\ \xi(\pi^{Z_{i,j}}x_j\ldots+\pi^k i_k) \\ \vdots \\ \xi x_n \end{pmatrix} \mathrm{d}\mathcal{X} \\ &= f(c_{i,j}i_k)\int_{\mathfrak{O}_\nu^n} f((\xi x_1,\ldots,\xi x_i,\ldots,\xi\pi^k i_k,\ldots,\xi x_n)^T)\mathrm{d}\mathcal{X}. \end{aligned}$$

We shall calculate this expression by splitting the space over which we are integrating with respect to each of the variables $x_i$ in turn. This will enable us to break the integral into parts we can solve.

That is,

$$\begin{aligned} I_1^{(\xi)}(i_k) &= f(c_{i,j}i_k)\int_{\mathfrak{O}_\nu^n} f((\xi x_1,\ldots,\xi x_i,\ldots,\xi\pi^k i_k,\ldots,\xi x_n)^T)\mathrm{d}\mathcal{X} \\ &= f(c_{i,j}i_k)\int_{x_1\in\mathfrak{O}_\nu\setminus\pi^{k+1}\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu^{n-1}} f((\xi x_1,\ldots,\xi x_n)^T)\mathrm{d}\mathcal{X} \\ &\quad + f(c_{i,j}i_k)\int_{\pi^{k+1}\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu^{n-1}} f((\xi x_2,\ldots,\xi\pi^k i_k,\ldots,\xi x_n)^T)\mathrm{d}\mathcal{X} \\ &= f(c_{i,j}i_k)\int_{x_1\in\mathfrak{O}_\nu\setminus\pi^{k+1}\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu^{n-1}} f((\xi x_1,\ldots,\xi x_n)^T)\mathrm{d}\mathcal{X} \\ &\quad + f(c_{i,j}i_k)\int_{\pi^{k+1}\mathfrak{O}_\nu}\int_{x_2\in\mathfrak{O}_\nu\setminus\pi^{k+1}\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu^{n-2}} f((\xi x_2,\ldots,\xi x_n)^T)\mathrm{d}\mathcal{X} \\ &\qquad + f(c_{i,j}i_k)\int_{(\pi^{k+1}\mathfrak{O}_\nu)^2}\int_{\mathfrak{O}_\nu^{n-2}} f((\xi x_3,\ldots,\xi\pi^k i_k,\ldots,\xi x_n)^T)\mathrm{d}\mathcal{X} \end{aligned}$$

repeating this procedure we eventually find:

$$\begin{aligned} &= f(c_{i,j}i_k)\sum_{l=1}^{j-1}\left\{\int_{(\pi^{k+1}\mathfrak{O}_\nu)^{l-1}}\int_{x_l\in\mathfrak{O}_\nu\setminus\pi^{k+1}\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu^{n-l}} f((\xi x_l,\ldots,\xi x_n)^T)\mathrm{d}\mathcal{X}\right\} \\ &\quad + f(c_{i,j}i_k)\int_{(\pi^{k+1}\mathfrak{O}_\nu)^{j-1}}\int_{\mathfrak{O}_\nu^{n-j+1}} f((\xi\pi^k i_k,\ldots,\xi x_n)^T)\mathrm{d}\mathcal{X}. \end{aligned}$$

Using the fact that the function $f$ is fundamental this is found to be,

$$I_1^{(\xi)}(i_k) = f(c_{i,j}i_k)\frac{(\rho^{k+1}-1)}{m}(j-1) + L^{(\xi)}(i_k),$$

where the integral $L^{(\xi)}(i_k)$ is defined to be,

$$L^{(\xi)}(i_k) := \ f(c_{i,j}i_k)\int_{\mathfrak{O}_\nu^{n-j+1}} f((\xi\pi^k i_k, \dots, \xi x_n)^T)\mathrm{d}\mathcal{X}.$$

In order to calculate the integral $L^{(\xi)}(i_k)$ we must now peel off the remaining variables $x_\ell$ where $\ell > j$ and further split this into parts we can solve. That is,

$$\begin{aligned}
L^{(\xi)}(i_k) &= f(c_{i,j}i_k)\int_{\mathfrak{O}_\nu}\int_{x_{j+1}\in\mathfrak{O}_\nu\setminus\pi^k\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu^{n-j-1}} f((\xi x_{j+1},\dots,\xi x_n)^T)\mathrm{d}\mathcal{X}\\
&\quad + f(c_{i,j}i_k)\int_{\mathfrak{O}_\nu}\int_{\pi^k\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu^{n-j-1}} f((\xi\pi^k i_k,\xi x_{j+2},\dots,\xi x_n)^T)\mathrm{d}\mathcal{X}\\
&= f(c_{i,j}i_k)\int_{\mathfrak{O}_\nu}\int_{x_{j+1}\in\mathfrak{O}_\nu\setminus\pi^k\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu^{n-j-1}} f((\xi x_{j+1},\dots,\xi x_n)^T)\mathrm{d}\mathcal{X}\\
&\quad + f(c_{i,j}i_k)\int_{\mathfrak{O}_\nu}\int_{\pi^k\mathfrak{O}_\nu}\int_{x_{j+2}\in\mathfrak{O}_\nu\setminus\pi^k\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu^{n-j-2}} f((\xi x_{j+2},\dots,\xi x_n)^T)\mathrm{d}\mathcal{X}\\
&\qquad + f(c_{i,j}i_k)\int_{\mathfrak{O}_\nu}\int_{(\pi^k\mathfrak{O}_\nu)^2}\int_{\mathfrak{O}_\nu^{n-j-2}} f((\xi\pi^k i_k,\xi x_{j+3},\dots,\xi x_n)^T)\mathrm{d}\mathcal{X}
\end{aligned}$$

If we repeat this procedure we eventually find,

$$\begin{aligned}
L^{(\xi)}(i_k) &= f(c_{i,j}i_k)\sum_{l=1}^{n-j}\left\{\int_{\mathfrak{O}_\nu}\int_{(\pi^k\mathfrak{O}_\nu)^{l-1}}\int_{x_{j+l}\in\mathfrak{O}_\nu\setminus\pi^k\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu^{n-j-l}} f((\xi x_{j+l},\dots,\xi x_n)^T)\mathrm{d}\mathcal{X}\right\}\\
&\quad + f(c_{i,j}i_k)f(\xi i_k)\int_{\mathfrak{O}_\nu}\int_{(\pi^k\mathfrak{O}_\nu)^{n-j}}\mathrm{d}\mathcal{X}.
\end{aligned}$$

Again by using the fact that $f$ is fundamental we are able to conclude that,

$$L^{(\xi)}(i_k) = f(c_{i,j}i_k)\frac{(\rho^k-1)}{m}(n-j) + f(c_{i,j}i_k)f(\xi i_k).$$

Finally, we have found that the integral $I_1^{(\xi)}(i_k)$ satisfies,

$$I_1^{(\xi)}(i_k) = f(c_{i,j}i_k)\frac{(\rho^{k+1}-1)}{m}(j-1) + f(c_{i,j}i_k)\frac{(\rho^k-1)}{m}(n-j) + f(c_{i,j}i_k)f(\xi i_k).$$

Since the previous calculation will be required throughout this chapter we shall state this result as a lemma.

**Lemma 6.3.1**

$$\begin{aligned}
K^{(\xi)}(i_k) &:= f(c_{i,j}i_k)\int_{\mathfrak{O}_\nu^n} f((\xi x_1,.., \overset{j^{th}\text{ position}}{\xi\pi^k i_k}, .., \xi x_n)^T)\mathrm{d}\mathcal{X},\\
&= f(c_{i,j}i_k)\frac{(\rho^{k+1}-1)}{m}(j-1) + f(c_{i,j}i_k)\frac{(\rho^k-1)}{m}(n-j) + f(c_{i,j}i_k)f(\xi i_k).
\end{aligned}$$

### The Integrals $I_2^{(\xi)}(i_k)$

In order to calculate the integrals $I_2^{(\xi)}(i_k)$ we shall require the substitutions,

$$x_i \longmapsto x_i + \pi^{-Z_{i,j}} c_{i,j}(\pi^{Z_{i,j}} x_j + \pi^{Z_{i,j}-1} i_{Z_{i,j}-1} + \ldots + \pi^k i_k) \qquad x_j \longmapsto \pi^{Z_{i,j}} x_j + \ldots + \pi^k i_k$$
$$\Rightarrow \mathrm{d}\mathcal{X} \longmapsto \rho^{-Z_{i,j}} \mathrm{d}\mathcal{X} \equiv \mathrm{d}\mathcal{X} \pmod{m}.$$

Having employed these substitutions we find that the integral $I_2^{(\xi)}(i_k)$ satisfies,

$$I_2^{(\xi)}(i_k) = \int_{\mathfrak{O}_\nu^n} f\begin{pmatrix} x_1 \\ \vdots \\ x_i + 2\pi^{-Z_{i,j}} c_{i,j}(\pi^{Z_{i,j}} x_j + \ldots + \pi^k i_k) \\ \vdots \\ (\pi^{Z_{i,j}} x_j \ldots + \pi^k i_k) \\ \vdots \\ x_n \end{pmatrix} f\begin{pmatrix} \xi x_1 \\ \vdots \\ \xi(x_i + 2\pi^{-Z_{i,j}} c_{i,j}(\ldots + \pi^k i_k)) \\ \vdots \\ \xi(\pi^{Z_{i,j}} x_j \ldots + \pi^k i_k) \\ \vdots \\ \xi x_n \end{pmatrix} \mathrm{d}\mathcal{X},$$

from which we may immediately deduce,

$$I_2^{(\xi)}(i_k) \ = \ f(2c_{i,j} i_k) f(\xi c_{i,j} i_k).$$

### Results for $dec_\nu(n_{i,j}, n_{i,j})$

Having calculated both the integrals $I_1^{(\xi)}(i_k)$ and $I_2^{(\xi)}(i_k)$ we are finally able to combine the results found in the previous two sections and evaluate the cocycle $dec_\nu$. That is,

$$\begin{aligned} dec_\nu(n_{i,j}, n_{i,j}) &= \prod_{k=0}^{Z_{i,j}-1} \prod_{\xi \in \mu_m} \xi^{\sum_{i_k} I_1^{(\xi)}(i_k)} \prod_{k=0}^{Z_{i,j}-1} \prod_{\xi \in \mu_m} \xi^{-\sum_{i_k} I_2^{(\xi)}(i_k)} \\ &= \Big(\prod_{\xi \in \mu_m} \xi\Big)^{\frac{(\rho-1)}{m}(j-1)\sum_{k=0}^{Z_{i,j}-1}(k+1)} \Big(\prod_{\xi \in \mu_m} \xi\Big)^{\frac{(\rho-1)}{m}(n-j)\sum_{k=0}^{Z_{i,j}-1}(k)} \\ &\quad \prod_{k=0}^{Z_{i,j}-1} \prod_{\xi \in \mu_m} \xi^{\sum_{i_k} f(c_{i,j} i_k) f(\xi i_k)} \prod_{k=0}^{Z_{i,j}-1} \prod_{\xi \in \mu_m} \xi^{-\sum_{i_k} f(2c_{i,j} i_k) f(\xi c_{i,j} i_k)} \end{aligned}$$

which, using Lemma 2.0.5, is found to be

$$\begin{aligned} &= (-1)^{\frac{(\rho-1)}{m}(j-1)\frac{Z_{i,j}(Z_{i,j}+1)}{2}} (-1)^{\frac{(\rho-1)}{m}(n-j)\frac{Z_{i,j}(Z_{i,j}-1)}{2}} . (c_{i,j}, \pi)_{\nu,m}^{Z_{i,j}} . (2, \pi)_{\nu,m}^{-Z_{i,j}} \\ &= (c_{i,j}/2, \pi)_{\nu,m}^{Z_{i,j}} \, (-1)^{\frac{(\rho-1)}{2^r}\frac{Z_{i,j}(Z_{i,j}+1)}{2}(j-1)} (-1)^{\frac{(\rho-1)}{2^r}\frac{Z_{i,j}(Z_{i,j}-1)}{2}(n-j)}, \end{aligned}$$

which completes the proof of our theorem.

$\square$

## 6.4 $dec_\nu$ on $N$ in $\mathrm{GL}_3(k_\nu)$

In this section we shall turn our attention towards finding some specific results for the cocycle $dec_\nu$ when restricted to $N$ in $\mathrm{GL}_3(k_\nu)$. Before we begin let us remind ourselves that in $\mathrm{GL}_3(k_\nu)$ we have defined the matrices,

$$n_{1,2} := \begin{pmatrix} 1 & \pi^{-z_{1,2}}c_{1,2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad n_{1,3} := \begin{pmatrix} 1 & 0 & \pi^{-z_{1,3}}c_{1,3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$n_{2,3} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \pi^{-z_{2,3}}c_{2,3} \\ 0 & 0 & 1 \end{pmatrix},$$

Using these matrices it is easily shown that in $\mathrm{GL}_3$ the general matrix $n \in N$ may be written as,

$$\begin{aligned} n :&= \begin{pmatrix} 1 & \pi^{-z_{1,2}}c_{1,2} & \pi^{-z_{1,3}}c_{1,3} \\ 0 & 1 & \pi^{-z_{2,3}}c_{2,3} \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \pi^{-z_{2,3}}c_{2,3} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \pi^{-z_{1,3}}c_{1,3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \pi^{-z_{1,2}}c_{1,2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= n_{2,3}n_{1,3}n_{1,2}. \end{aligned}$$

Furthermore the matrices $n_{i,j}$ also satisfy,

$$n_{1,2}n_{1,3} = \begin{pmatrix} 1 & \pi^{-z_{1,2}}c_{1,2} & \pi^{-z_{1,3}}c_{1,3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = n_{1,3}n_{1,2}$$

$$n_{2,3}n_{1,3} = \begin{pmatrix} 1 & 0 & \pi^{-z_{1,3}}c_{1,3} \\ 0 & 1 & \pi^{-z_{2,3}}c_{2,3} \\ 0 & 0 & 1 \end{pmatrix} = n_{1,3}n_{2,3}$$

$$n_{2,3}n_{1,2} = \begin{pmatrix} 1 & \pi^{-z_{1,2}}c_{1,2} & 0 \\ 0 & 1 & \pi^{-z_{2,3}}c_{2,3} \\ 0 & 0 & 1 \end{pmatrix},$$

$$n_{1,2}n_{2,3} = \begin{pmatrix} 1 & \pi^{-z_{1,2}}c_{1,2} & \pi^{-(z_{1,2}+z_{2,3})}c_{1,2}c_{2,3} \\ 0 & 1 & \pi^{-z_{2,3}}c_{2,3} \\ 0 & 0 & 1 \end{pmatrix}.$$

In this section we shall concentrate on calculating the following results:

$$dec_\nu(n_{2,3}, n_{1,3}), \quad dec_\nu(n_{1,3}, n_{1,2}), \quad dec_\nu(n_{2,3}, n_{1,2}), \quad dec_\nu(n_{2,3}n_{1,3}, n_{1,2}).$$

Let us for a moment assume that the cocycle $dec_\nu$ does indeed split on $N$. Then, considering the relations for the matrices $n_{i,j}$ given above, these four results will enable us to completely describe this splitting for $\mathrm{GL}_3(k_\nu)$. Once we have this splitting we shall then be able to recover $dec_\nu(n_1, n_2)$ for any $n_1,\ n_2 \in N$ in $\mathrm{GL}_3(k_\nu)$.

In order to find these results we shall require the dissection of $\mathfrak{O}_\nu^3$ described in the section 6.2.1 when $n = 3$. Then, as with the previous chapter, we shall once again concentrate on finding the integrals $I_1^{(\epsilon)}(i_k)$ and $I_2^{(\epsilon)}(i_k)$ with $i_k$ strictly non-zero.

**Theorem 6.4.1** *For each,*

$$n_{2,3} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \pi^{-Z_{2,3}}c_{2,3} \\ 0 & 0 & 1 \end{pmatrix}, \quad n_{1,3} := \begin{pmatrix} 1 & 0 & \pi^{-Z_{1,3}}c_{1,3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in N,$$

*the cocycle $dec_\nu$ satisfies,*

- $Z_{1,3}, Z_{2,3} \leq 0$ :
$$dec_\nu(n_{2,3}, n_{1,3}) = 1$$
- $0 < Z_{1,3} < Z_{2,3}$ :
$$dec_\nu(n_{2,3}, n_{1,3}) = (c_{1,3}, \pi)_{\nu,m}^{Z_{1,3}}$$
- $0 < Z_{2,3} \leq Z_{1,3}$ :
$$dec_\nu(n_{2,3}, n_{1,3}) = (c_{2,3}, \pi)_{\nu,m}^{Z_{2,3}}.$$

**PROOF OF THEOREM:**

In order to prove this theorem we shall once again begin by calculating the integrals $I_1^{(\epsilon)}(i_k)$ and $I_2^{(\epsilon)}(i_k)$ in each of the possible cases which arise. Using these calculations we shall then be able to construct results for the cocycle $dec_\nu$.

Before we begin this proof, by Theorems 6.2.1 and 6.2.2, we may already assume that if either $Z_{2,3} < 0,\ Z_{1,3} \leq 0$ we have,

$$dec_\nu(n_{2,3}, n_{1,3}) \ = \ 1.$$

Therefore, for the remainder of this proof we shall assume $Z_{2,3} \geq 0$ and $Z_{1,3} > 0$.

### The Integrals $I_1^{(\xi)}(i_k)$

Since $Z_{1,3} > 0$ we are free to use the dissection given in Section 6.2.1. Therefore, employing the substitutions,

$$x_3 \longmapsto \pi^{Z_{1,3}} x_3 + \pi^{Z_{1,3}-1} i_{Z_{1,3}-1} + \ldots + \pi^k i_k \quad \Rightarrow \quad \mathrm{d}x_3 \longmapsto |\pi^{Z_{1,3}}|_\nu \mathrm{d}x_3 \equiv \mathrm{d}x_3 \pmod m,$$

we are able to write,

$$I_1^{(\xi)}(i_k) = \int_{\mathfrak{O}_\nu^3} f\begin{pmatrix} x_1 \\ x_2+\pi^{-Z_{2,3}} c_{2,3}(\pi^{Z_{1,3}}x_3+\ldots+\pi^k i_k) \\ (\pi^{Z_{1,3}}x_3\ldots+\pi^k i_k) \end{pmatrix} f\begin{pmatrix} \xi x_1 \\ \xi x_2 \\ \xi(\pi^{Z_{1,3}}x_3\ldots+\pi^k i_k) \end{pmatrix} \mathrm{d}\mathcal{X}. \tag{6.1}$$

In order to calculate this integral we shall have to consider two possible distinct cases depending on the value of $k$.

**Case (1) : $k < Z_{2,3}$ :**

Whenever $k < Z_{2,3}$ this integral becomes,

$$I_1^{(\xi)}(i_k) = f(c_{2,3} i_k) \int_{\mathfrak{O}_\nu^3} f((\xi x_1, \xi x_2, \xi \pi^k i_k)^T) \mathrm{d}\mathcal{X}.$$

By Lemma 6.3.1 on page 151 we see that, by taking $i = 2,\ j = n = 3$, we immediately find

$$I_1^{(\xi)}(i_k) = f(c_{2,3} i_k) f(\xi i_k),$$

and therefore, whenever $k < Z_{2,3}$, we have

$$\prod_{\xi \in \mu_m} \xi^{\sum_{i_k} I_1^{(\xi)}(i_k)} = (c_{2,3}, \pi)_{\nu,m}.$$

**Case (2) : $k \geq Z_{2,3}$ :**

Let us now consider the case when $k \geq Z_{2,3}$. We should be aware that this case is only a possibility when we have $Z_{2,3} < Z_{1,3}$. Unlike the previous case we shall have to return to equation (6.1) and calculate,

$$I_1^{(\xi)}(i_k) = \int_{\mathfrak{O}_\nu^3} f\begin{pmatrix} x_1 \\ x_2+\pi^{k-Z_{2,3}} c_{2,3} i_k+\ldots \\ \pi^k i_k \end{pmatrix} f\begin{pmatrix} \xi x_1 \\ \xi x_2 \\ \xi \pi^k i_k \end{pmatrix} \mathrm{d}\mathcal{X}.$$

To solve this we shall split the space over which we are integrating into six parts and then calculate the integral over each separately.

After some consideration we eventually find,

$$I_1^{(\xi)}(i_k) = \int_{\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu\setminus\pi^{k-Z_{2,3}}\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu} f((x_1,x_2))f(\xi(x_1,x_2))\mathrm{d}\mathcal{X} \tag{1}$$

$$+\int_{\mathfrak{O}_\nu\setminus\pi^{k-Z_{2,3}+1}\mathfrak{O}_\nu}\int_{\pi^{k-Z_{2,3}}\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu} f(x_1)f(\xi x_1)\mathrm{d}\mathcal{X} \tag{2}$$

$$+\int_{\pi^{k-Z_{2,3}+1}\mathfrak{O}_\nu}\int_{\pi^{k-Z_{2,3}}\mathfrak{O}_\nu\setminus\pi^{k-Z_{2,3}+1}\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu} f\begin{pmatrix} x_1 \\ x_2+\pi^{k-Z_{2,3}}c_{2,3}i_k+\dots \\ \pi^k i_k\end{pmatrix} f(\xi x_2)\mathrm{d}\mathcal{X} \tag{3}$$

$$+f(c_{2,3}i_k)\int_{\pi^{k-Z_{2,3}+1}\mathfrak{O}_\nu\setminus\pi^{k+1}\mathfrak{O}_\nu}\int_{\pi^{k-Z_{2,3}+1}\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu} f(\xi(x_1,x_2))d(x_1,x_2)\mathrm{d}x_3 \tag{4}$$

$$+f(c_{2,3}i_k)\int_{\pi^{k+1}\mathfrak{O}_\nu}\int_{\pi^{k-Z_{2,3}+1}\mathfrak{O}_\nu\setminus\pi^{k+1}\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu} f(\xi x_2)\mathrm{d}\mathcal{X} \tag{5}$$

$$+\int_{\pi^{k+1}\mathfrak{O}_\nu}\int_{\pi^{k+1}\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu} f(c_{2,3}i_k)f(\xi i_k)\mathrm{d}\mathcal{X} \tag{6}.$$

We shall now discuss this expression term by term.

Since we have chosen the function $f$ such that $f.f\xi = 0$ we immediately find that integrals (1) and (2) are trivial. For (4) and (5) we notice that neither of these have any dependence on $\xi$. Therefore, using Lemma 5.2.3, we need only consider these terms modulo 2 in the case that $m = 2^r$. So, assuming $m = 2^r$ and using the fact that $f$ is fundamental, (4) and (5) are found to be congruent modulo $m$ and independent of $\xi$. Since there are two of them their sum must in fact be congruent to zero modulo 2 and we may assume that the terms cancel.

So we are simply left with the integrals (3) and (6). That is,

$$I_1^{(\xi)}(i_k) = L^{(\xi)}(i_k) + f(c_{2,3}i_k)f(\xi i_k).$$

where the integral $L^{(\xi)}(i_k)$ is defined to be,

$$L^{(\xi)}(i_k) = \int_{\pi^{k-Z_{2,3}+1}\mathfrak{O}_\nu}\int_{\pi^{k-Z_{2,3}}\mathfrak{O}_\nu\setminus\pi^{k-Z_{2,3}+1}\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu} f\begin{pmatrix} x_1 \\ x_2+\pi^{k-Z_{2,3}}c_{2,3}i_k+. \\ \pi^k i_k\end{pmatrix} f(\xi x_2)\mathrm{d}\mathcal{X}.$$

To calculate this we shall once again split this up into various integrals that we can solve. Firstly, having performed a simple change of variables this expression becomes,

$$\begin{aligned}
L^{(\xi)}(i_k) &= \int_{\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu\setminus\pi\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu} f\begin{pmatrix} \pi x_1 \\ (x_2+c_{2,3}i_k)+\pi c_{2,3}i_{k+1}+\dots+\pi^{Z_{1,3}-k}c_{2,3}x_3 \\ \pi^{Z_{2,3}}i_k\end{pmatrix} f(\xi x_2)\mathrm{d}\mathcal{X} \\
&= \int_{\mathfrak{O}_\nu}\underset{(x_2+c_{2,3}i_k)\not\equiv 0(\pi)}{\int_{\mathfrak{O}_\nu\setminus\pi\mathfrak{O}_\nu}}\int_{\mathfrak{O}_\nu} f(x_2+c_{2,3}i_k)f(\xi x_2)\mathrm{d}\mathcal{X} \\
&\quad + \int_{\mathfrak{O}_\nu}\underset{(x_2+c_{2,3}i_k)\equiv 0(\pi)}{\int_{\mathfrak{O}_\nu\setminus\pi\mathfrak{O}_\nu}}\int_{\mathfrak{O}_\nu} f\begin{pmatrix} \pi x_1 \\ (x_2+c_{2,3}i_k)+\pi c_{2,3}i_{k+1}+\dots+\pi^{Z_{1,3}-k}c_{2,3}x_3 \\ \pi^{Z_{2,3}}i_k\end{pmatrix} f(\xi x_2)\mathrm{d}\mathcal{X} \\
&=: L_1^{(\xi)}(i_k) + L_2^{(\xi)}(i_k) \qquad \text{respectively.}
\end{aligned}$$

Using Lemma 5.5.1 the first integral $L_1^{(\xi)}(i_k)$ is found to satisfy,

$$\sum_{i_k \in \Pi \setminus 0} L_1^{(\xi)}(i_k) = \frac{(\rho - 1)}{m}.$$

For the second integral we must use the substitution,

$$x_2 \longmapsto -c_{2,3} i_k + \pi x_2 \quad \Rightarrow \quad \mathrm{d}x_2 \longmapsto |\pi|_\nu \mathrm{d}x_2 \equiv \mathrm{d}x_2 \quad (\text{mod } m),$$

which allows us to write,

$$L_2^{(\xi)}(i_k) = f(\xi(-c_{2,3} i_k)) \int_{\mathfrak{O}_\nu^3} f\begin{pmatrix} \pi x_1 \\ \pi(x_2 + c_{2,3} i_{k+1}) + \ldots + \pi^{Z_{1,3}-k} c_{2,3} x_3 \\ \pi^{Z_{2,3}} i_k \end{pmatrix} \mathrm{d}\mathcal{X}.$$

Finally, by making the change of variable

$$x_2 \longmapsto x_2 - c_{2,3} i_{k+1} - \ldots - \pi^{Z_{1,3}-k-1} c_{2,3} x_3, \qquad \Rightarrow \qquad \mathrm{d}\mathcal{X} \longmapsto \mathrm{d}\mathcal{X} \quad (\text{mod } m),$$

and using the calculation in Lemma 6.3.1 we are able to write,

$$\begin{aligned} L_2^{(\xi)}(i_k) &= f(\xi(-c_{2,3} i_k)) \int_{\mathfrak{O}_\nu^3} f((\pi x_1, \pi x_2, \pi^{Z_{2,3}} i_k)^T) \mathrm{d}\mathcal{X} \\ &= f(i_k) f(\xi(-c_{2,3} i_k)). \end{aligned}$$

We are now in a position to conclude that,

$$I_1^{(\xi)}(i_k) = f(c_{2,3} i_k) f(\xi i_k) + \frac{(\rho - 1)}{m} + f(i_k) f(\xi(-c_{2,3} i_k)).$$

Therefore, using Lemma 2.0.5, whenever $k \geq Z_{2,3}$ we find

$$\prod_{\xi \in \mu_m} \xi^{\sum_{i_k} I_1^{(\xi)}(i_k)} = (c_{2,3}, \pi)_{\nu,m} . (-1)^{\frac{(\rho-1)}{m}} . (-c_{2,3}, \pi)_{\nu,m}^{-1} = 1.$$

### The Integrals $I_2^{(\xi)}(i_k)$

In order to calculate the integrals $I_2^{(\xi)}(i_k)$ we again use the dissection given in Section 6.2.1. For this integral we employ the substitutions,

$$x_1 \longmapsto x_1 + \pi^{-Z_{1,3}} c_{1,3}(\pi^{Z_{1,3}} x_3 + \pi^{Z_{1,3}-1} i_{Z_{1,3}-1} + \ldots + \pi^k i_k) \qquad x_3 \longmapsto \pi^{Z_{1,3}} x_3 + \ldots + \pi^k i_k$$
$$\Rightarrow \mathrm{d}\mathcal{X} \longmapsto \rho^{-Z_{1,3}} \mathrm{d}\mathcal{X} \equiv \mathrm{d}\mathcal{X} \pmod{m}.$$

Using these we find,

$$I_2^{(\xi)}(i_k) = \int_{\mathfrak{O}_\nu^3} f\begin{pmatrix} x_1+\pi^{-Z_{1,3}}c_{1,3}(\pi^{Z_{1,3}}x_3+\ldots+\pi^k i_k) \\ x_2+\pi^{-Z_{2,3}}c_{2,3}(\pi^{Z_{1,3}}x_3+\ldots+\pi^k i_k) \\ (\pi^{Z_{1,3}}x_3+\ldots+\pi^k i_k) \end{pmatrix} f\begin{pmatrix} \xi(x_1+\pi^{-Z_{1,3}}c_{1,3}(\ldots+\pi^k i_k)) \\ \xi x_2 \\ \xi(\pi^{Z_{1,3}}x_3+\ldots+\pi^k i_k) \end{pmatrix} \mathrm{d}\mathcal{X},$$

which clearly splits into the following two cases,

$$= \begin{cases} f(c_{1,3} i_k) f(\xi c_{1,3} i_k) = 0 & Z_{1,3} \geq Z_{2,3} \\ f(c_{2,3} i_k) f(\xi c_{1,3} i_k) & Z_{1,3} < Z_{2,3}. \end{cases}$$

Using these results together with Lemma 2.0.5 we are able to conclude that,

$$Z_{2,3} \leq Z_{1,3}: \qquad \prod_{\xi \in \mu_m} \xi^{\sum_{i_k} - I_2^{(\xi)}(i_k)} = 1. \qquad\qquad Z_{2,3} > Z_{1,3}: \qquad \prod_{\xi \in \mu_m} \xi^{\sum_{i_k} - I_2^{(\xi)}(i_k)} = (c_{1,3}/c_{2,3}, \pi)_{\nu,m}.$$

Considering the calculations of the integrals $I_1^{(\xi)}(i_k)$ and $I_2^{(\xi)}(i_k)$ given in the previous two sections we shall have three distinct cases when evaluating,

$$dec_\nu(n_{2,3}, n_{1,3}) = \prod_{k=0}^{Z_{1,3}-1} \prod_{\xi \in \mu_m} \xi^{\sum_{i_k} I_1^{(\xi)}(i_k)} \prod_{k=0}^{Z_{1,3}-1} \prod_{\xi \in \mu_m} \xi^{-\sum_{i_k} I_2^{(\xi)}(i_k)}.$$

**Results for $0 < Z_{1,3} < Z_{2,3}$ :**

$$dec_\nu(n_{2,3}, n_{1,3}) = (c_{2,3}, \pi)_{\nu,m}^{Z_{1,3}} . (c_{1,3}/c_{2,3}, \pi)_{\nu,m}^{Z_{1,3}}$$
$$= (c_{1,3}, \pi)_{\nu,m}^{Z_{1,3}}.$$

**Results for $0 < Z_{1,3} = Z_{2,3}$ :**

$$dec_\nu(n_{2,3}, n_{1,3}) = (c_{2,3}, \pi)_{\nu,m}^{Z_{1,3}} .1$$
$$= (c_{2,3}, \pi)_{\nu,m}^{Z_{2,3}} \qquad \text{since } Z_{1,3} = Z_{2,3}.$$

**Results for $0 \leq Z_{2,3} < Z_{1,3}$ :**

For this last result we must split the product over $k$ of the integrals $I_1^{(\xi)}(i_k)$. Therefore in this case we find,

$$
\begin{aligned}
dec_\nu(n_{2,3}, n_{1,3}) &= \prod_{k=0}^{Z_{1,3}-1} \prod_{\xi \in \mu_m} \xi^{\sum_{i_k} I_1^{(\xi)}(i_k)} \prod_{k=0}^{Z_{1,3}-1} \prod_{\xi \in \mu_m} \xi^{-\sum_{i_k} I_2^{(\xi)}(i_k)} \\
&= \prod_{k=0}^{Z_{2,3}-1} \cdot \prod_{k=Z_{2,3}}^{Z_{1,3}-1} \prod_{\xi \in \mu_m} \xi^{\sum_{i_k} I_1^{(\xi)}(i_k)} \prod_{k=0}^{Z_{1,3}-1} \prod_{\xi \in \mu_m} \xi^{-\sum_{i_k} I_2^{(\xi)}(i_k)} \\
&= (c_{2,3}, \pi)_{\nu,m}^{Z_{2,3}}.
\end{aligned}
$$

Finally, putting these three cases together, we do indeed find the results as stated in the theorem.

□

**Theorem 6.4.2** *For each,*

$$
n_{1,3} := \begin{pmatrix} 1 & 0 & \pi^{-Z_{1,3}} c_{1,3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad n_{1,2} := \begin{pmatrix} 1 & \pi^{-Z_{1,2}} c_{1,2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in N,
$$

*the cocycle $dec_\nu$ satisfies,*

- $Z_{1,3}, Z_{1,2} \leq 0$ :
$$dec_\nu(n_{1,3}, n_{1,2}) = 1$$
- $0 < Z_{1,3} < Z_{1,2}$ :
$$dec_\nu(n_{1,3}, n_{1,2}) = (c_{1,3}, \pi)_{\nu,m}^{Z_{1,3}}$$
- $0 < Z_{1,2} \leq Z_{1,3}$ :
$$dec_\nu(n_{1,3}, n_{1,2}) = (c_{1,2}, \pi)_{\nu,m}^{Z_{1,2}} (-1)^{\frac{(\rho-1)}{2^r} Z_{1,2}}.$$

**PROOF OF THEOREM:**

Using Theorems 6.2.1 and 6.2.2, whenever either $Z_{1,3} < 0$ or $Z_{1,2} \leq 0$, the cocycle satisfies

$$dec_\nu(n_{1,3}, n_{1,2}) = 1.$$

Therefore, for the remainder of this proof let us assume that $Z_{1,3} \geq 0$ and $Z_{1,2} > 0$.

### The Integrals $I_1^{(\xi)}(i_k)$

Since we are assuming that $Z_{1,2} > 0$ we are able to use the substitutions,

$$x_2 \longmapsto \pi^{Z_{1,2}}x_2 + \pi^{Z_{1,2}-1}i_{Z_{1,2}-1} + \ldots + \pi^k i_k \quad \Rightarrow \quad \mathrm{d}x_2 \longmapsto |\pi^{Z_{1,2}}|_\nu \mathrm{d}x_2 \equiv \mathrm{d}x_2 \pmod{m}.$$

Using these we are able to write,

$$\begin{aligned}
I_1^{(\xi)}(i_k) &= \int_{\mathfrak{O}_\nu^3} f\begin{pmatrix} x_1+\pi^{-Z_{1,3}}c_{1,3}x_3 \\ (\pi^{Z_{1,2}}x_2\ldots+\pi^k i_k) \\ x_3 \end{pmatrix} f\begin{pmatrix} \xi x_1 \\ \xi(\pi^{Z_{1,2}}x_2\ldots+\pi^k i_k) \\ \xi x_3 \end{pmatrix} \mathrm{d}\mathcal{X} \\
&= \int_{\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu\setminus\pi^{Z_{1,3}}\mathfrak{O}_\nu} f(c_{1,3}x_3) f\begin{pmatrix} \xi x_1 \\ \xi\pi^k i_k \\ \xi x_3 \end{pmatrix} \mathrm{d}\mathcal{X} + \int_{\mathfrak{O}_\nu^3} f\begin{pmatrix} x_1+c_{1,3}x_3 \\ \ldots+\pi^k i_k \\ \pi^{Z_{1,3}}x_3 \end{pmatrix} f\begin{pmatrix} \xi x_1 \\ \xi\pi^k i_k \\ \xi\pi^{Z_{1,3}}x_3 \end{pmatrix} \mathrm{d}\mathcal{X} \\
&=: L_1^{(\xi)}(i_k) + L_2^{(\xi)}(i_k) \qquad \text{defined respectively.} \qquad (6.2)
\end{aligned}$$

where, for the integral $L_2^{(\xi)}(i_k)$ we have already used the change of variable, $x_3 \longmapsto \pi^{Z_{1,3}}x_3$.

In order to calculate the integrals $L_1^{(\xi)}(i_k)$ and $L_2^{(\xi)}(i_k)$ we shall have to split this into two possible cases depending on the value of $k$.

**Case (1)** $\quad k \geq Z_{1,3}$ :

Let us note that this case is only possible when $Z_{1,3} < Z_{1,2}$. Assuming this is so, for the integral $L_1^{(\xi)}(i_k)$, we find

$$L_1^{(\xi)}(i_k) = \int_{\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu\setminus\pi^{Z_{1,3}}\mathfrak{O}_\nu} f(c_{1,3}x_3) f\begin{pmatrix} \xi x_1 \\ \xi x_3 \end{pmatrix} \mathrm{d}\mathcal{X}.$$

Since this integral has no dependence on $i_k$, by Lemma 5.2.2, we need consider it no further and may simply disregard it.

Let us now consider the integral $L_2^{(\xi)}(i_k)$. In order to calculate this we shall begin by splitting the space over which we are integrating into two parts depending on the first variable $x_1$. We shall then calculate each of these integrals separately. That is,

$$\begin{aligned}
L_2^{(\xi)}(i_k) = &\int_{\mathfrak{O}_\nu\setminus\pi^{k+1}\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu^2} f\begin{pmatrix} x_1+c_{1,3}x_3 \\ \ldots+\pi^k i_k \\ \pi^{Z_{1,3}}x_3 \end{pmatrix} f\begin{pmatrix} \xi x_1 \\ \xi\pi^{Z_{1,3}}x_3 \end{pmatrix} \mathrm{d}\mathcal{X} \\
&+ \int_{\pi^{k+1}\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu^2} f\begin{pmatrix} x_1+c_{1,3}x_3 \\ \ldots+\pi^k i_k \\ \pi^{Z_{1,3}}x_3 \end{pmatrix} f\begin{pmatrix} \xi\pi^k i_k \\ \xi\pi^{Z_{1,3}}x_3 \end{pmatrix} \mathrm{d}\mathcal{X} \\
=: &\, L_3^{(\xi)}(i_k) + L_4^{(\xi)}(i_k).
\end{aligned}$$

We begin by considering $L_3^{(\xi)}(i_k)$ and once again splitting the space over which we are integrating. If we again neglect the terms independent of $i_k$ and use the fact that $f$ is fundamental then this integral becomes,

$$
\begin{aligned}
L_3^{(\xi)}(i_k) &= \int_{\mathfrak{O}_\nu \setminus \pi^{k+1}\mathfrak{O}_\nu} \int_{\mathfrak{O}_\nu} \int_{\pi^{k-Z_{1,3}}\mathfrak{O}_\nu} f\left(\tfrac{x_1+c_{1,3}x_3}{\pi^k i_k}\right) f(\xi x_1) \mathrm{d}\mathcal{X} \\
&= f(i_k) \int_{\mathfrak{O}_\nu \setminus \pi^{Z_{1,3}+1}\mathfrak{O}_\nu} \int_{\mathfrak{O}_\nu} \int_{\pi^{k-Z_{1,3}}\mathfrak{O}_\nu \cap B(-\frac{x_1}{c_{1,3}}, \pi^{-(k+1)})} f(\xi x_1) \mathrm{d}\mathcal{X} \\
&= f(i_k) \int_{\pi^{k-Z_{1,3}}\mathfrak{O}_\nu \setminus \pi^{k+1}\mathfrak{O}_\nu} \int_{\mathfrak{O}_\nu} \int_{\pi^{k-Z_{1,3}}\mathfrak{O}_\nu} f(\xi x_1) \mathrm{d}\mathcal{X} \\
&= f(i_k) \frac{(\pi^{Z_{1,3}+1} - 1)}{m}.
\end{aligned}
$$

Let us now consider the integral $L_4^{(\xi)}(i_k)$. Having neglected the terms independent of $i_k$ this integral becomes,

$$
\begin{aligned}
L_4^{(\xi)}(i_k) &= f(\xi i_k) \int_{\pi^{k+1}\mathfrak{O}_\nu} \int_{\mathfrak{O}_\nu} \int_{\pi^{k-Z_{1,3}}\mathfrak{O}_\nu} f\left(\tfrac{x_1+c_{1,3}x_3}{\pi^k i_k}\right) \mathrm{d}\mathcal{X} \\
&= f(\xi i_k) \int_{\pi^{k+1}\mathfrak{O}_\nu} \int_{\mathfrak{O}_\nu} \int_{\pi^{k-Z_{1,3}}\mathfrak{O}_\nu \setminus \pi^{k+1}\mathfrak{O}_\nu} f(c_{1,3}x_3) \mathrm{d}\mathcal{X} + f(i_k) f(\xi i_k) \int_{\pi^{k+1}\mathfrak{O}_\nu} \int_{\mathfrak{O}_\nu} \int_{\pi^{k+1}\mathfrak{O}_\nu} \mathrm{d}\mathcal{X} \\
&= f(i_k) \frac{(\pi^{Z_{1,3}+1} - 1)}{m}.
\end{aligned}
$$

In conclusion, whenever $k \geq Z_{1,3}$, we have found that the integral $I_1^{(\xi)}(i_k)$ satisfies,

$$
I_1^{(\xi)}(i_k) = 2 f(i_k) \frac{(\pi^{Z_{1,3}+1} - 1)}{m} = 0.
$$

That is, since these expressions have no dependence on $\xi$, by Lemma 5.2.3, we need only consider them modulo 2. Therefore we have found that,

$$
\prod_{\xi \in \mu_m} \xi^{\sum_{i_k} I_1^{(\xi)}(i_k)} = 1.
$$

Case (2) $\quad k < Z_{1,3}$ :

Returning to equation (6.2), whenever $Z_{1,3} > k$, we must once again calculate

$$
I_1^{(\xi)}(i_k) =: L_1^{(\xi)}(i_k) + L_2^{(\xi)}(i_k).
$$

We begin by considering the integral $L_1^{(\xi)}(i_k)$. To calculate this we shall once again split the integral and ignore the terms independent of $i_k$. Then, using the fact that $f$ is fundamental, we shall have

$$\begin{aligned}
L_1^{(\xi)}(i_k) &= \int_{\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu \backslash \pi^{Z_{1,3}}\mathfrak{O}_\nu} f(c_{1,3}x_3) f\begin{pmatrix} \xi x_1 \\ \xi\pi^k i_k \\ \xi x_3 \end{pmatrix} \mathrm{d}\mathcal{X} \\
&= f(\xi i_k) \int_{\pi^{k+1}\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu}\int_{\pi^k\mathfrak{O}_\nu \backslash \pi^{Z_{1,3}}\mathfrak{O}_\nu} f(c_{1,3}x_3)\mathrm{d}\mathcal{X} \\
&= f(\xi i_k)\frac{(\pi^{Z_{1,3}-k}-1)}{m}.
\end{aligned}$$

For the second integral $L_2^{(\xi)}(i_k)$, we begin by splitting with respect to the first variable $x_1$. Then we find,

$$\begin{aligned}
L_2^{(\xi)}(i_k) &= \int_{\mathfrak{O}_\nu^3} f\begin{pmatrix} x_1+c_{1,3}x_3 \\ \pi^k i_k \end{pmatrix} f\begin{pmatrix} \xi x_1 \\ \xi\pi^k i_k \end{pmatrix} \mathrm{d}\mathcal{X} \\
&= \int_{\mathfrak{O}_\nu \backslash \pi^{k+1}\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu} f\begin{pmatrix} x_1+c_{1,3}x_3 \\ \pi^k i_k \end{pmatrix} f(\xi x_1)\mathrm{d}\mathcal{X} \\
&\quad + f(\xi i_k)\int_{\pi^{k+1}\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu \backslash \pi^{k+1}\mathfrak{O}_\nu} f(c_{1,3}x_3)\mathrm{d}\mathcal{X} \\
&\quad\quad + f(i_k)f(\xi i_k)\int_{\pi^{k+1}\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu}\int_{\pi^{k+1}\mathfrak{O}_\nu} \mathrm{d}\mathcal{X},
\end{aligned}$$

where clearly this last integral is trivial since $f.f\xi = 0$. For the first integral we must again consider the space over which this expression will have some dependence on $i_k$. That is,

$$\begin{aligned}
L_2^{(\xi)}(i_k) &= f(i_k)\int_{\mathfrak{O}_\nu \backslash \pi^{k+1}\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu}\int_{-\frac{x_1}{c_{1,3}}+\pi^{k+1}\mathfrak{O}_\nu} f(\xi x_1)\mathrm{d}\mathcal{X} \\
&\quad + f(\xi i_k)\int_{\pi^{k+1}\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu \backslash \pi^{k+1}\mathfrak{O}_\nu} f(c_{1,3}x_3)\mathrm{d}\mathcal{X} \\
&= f(i_k)\frac{(\pi^{k+1}-1)}{m} + f(\xi i_k)\frac{(\pi^{k+1}-1)}{m} = 0,
\end{aligned}$$

since these expressions have no dependence on $\xi$ and, by Lemma 5.2.3, need only be considered modulo 2.

So, whenever $k < Z_{1,3}$, we have found that the integral $I_1^{(\xi)}(i_k)$ satisfies,

$$I_1^{(\xi)}(i_k) = L_1^{(\xi)}(i_k) = f(\xi i_k)\frac{(\pi^{Z_{1,3}-k}-1)}{m}.$$

Therefore we have found,

$$\prod_{\xi\in\mu_m} \xi^{\sum_{i_k} I_1^{(\xi)}(i_k)} = (-1)^{\frac{(\rho-1)}{2^r}(Z_{1,3}-k)}.$$

**The Integrals $I_2^{(\xi)}(i_k)$**

In order to calculate these integrals we shall require the substitutions,

$$x_1 \longmapsto x_1 + \pi^{-Z_{1,2}} c_{1,2}(\pi^{Z_{1,2}} x_2 + \pi^{Z_{1,2}-1} i_{Z_{1,2}-1} + \dots + \pi^k i_k) \qquad x_2 \longmapsto \pi^{Z_{1,2}} x_2 + \dots + \pi^k i_k$$
$$\Rightarrow \mathrm{d}\mathcal{X} \longmapsto \rho^{-Z_{1,2}} \mathrm{d}\mathcal{X} \equiv \mathrm{d}\mathcal{X} \pmod{m}.$$

Using these we are able to write,

$$\begin{aligned} I_2^{(\xi)}(i_k) &= \int_{\mathfrak{D}_\nu^3} f\begin{pmatrix} x_1+\pi^{-Z_{1,2}}c_{1,2}(\pi^{Z_{1,2}}x_2+\dots+\pi^k i_k)+\pi^{-Z_{1,3}}c_{1,3}x_3 \\ (\pi^{Z_{1,2}}x_2+\dots+\pi^k i_k) \\ x_3 \end{pmatrix} f\begin{pmatrix} \xi(x_1+\pi^{-Z_{1,2}}c_{1,2}(\dots+\pi^k i_k)) \\ \xi(\pi^{Z_{1,2}}x_2+\dots+\pi^k i_k) \\ \xi x_3 \end{pmatrix} \mathrm{d}\mathcal{X} \\ &= f(\xi c_{1,2} i_k) \int_{\mathfrak{D}_\nu^3} f\begin{pmatrix} x_1+\pi^{-Z_{1,2}}c_{1,2}(\pi^{Z_{1,2}}x_2+\dots+\pi^k i_k)+\pi^{-Z_{1,3}}c_{1,3}x_3 \\ (\pi^{Z_{1,2}}x_2+\dots+\pi^k i_k) \\ x_3 \end{pmatrix} \mathrm{d}\mathcal{X}. \end{aligned} \tag{6.3}$$

Considering this expression we shall begin by looking at two distinct cases.

**Case (1)** $\quad k < Z_{1,2} - Z_{1,3}$ :

Let us begin by noting that this case is only possible when $Z_{1,2} > Z_{1,3}$. Assuming this is so, whenever $k < Z_{1,2} - Z_{1,3}$, equation (6.3) simply becomes,

$$I_2^{(\xi)}(i_k) = f(c_{1,2}i_k)f(\xi c_{1,2}i_k) = 0, \qquad \text{since } f.f\xi = 0.$$

**Case (2)** $\quad k \geq Z_{1,2} - Z_{1,3}$ :

For the second of the two cases, when $k \geq Z_{1,2} - Z_{1,3}$, we shall have a little more work to do. Equation (6.3) now becomes,

$$\begin{aligned} I_2^{(\xi)}(i_k) = {} & f(\xi c_{1,2}i_k) \int_{\mathfrak{D}_\nu} \int_{\mathfrak{D}_\nu} \int_{\mathfrak{D}_\nu \setminus \pi^{k-(Z_{1,2}-Z_{1,3})}\mathfrak{D}_\nu} f(c_{1,3}x_3)\mathrm{d}\mathcal{X} \\ & + f(\xi c_{1,2}i_k) \int_{\mathfrak{D}_\nu} \int_{\mathfrak{D}_\nu} \int_{\pi^{k-(Z_{1,2}-Z_{1,3})}\mathfrak{D}_\nu \setminus \pi^{k-(Z_{1,2}-Z_{1,3})+1}\mathfrak{D}_\nu} f\begin{pmatrix} x_1+\pi^{-Z_{1,2}}c_{1,2}(.+\pi^k i_k)+\pi^{-Z_{1,3}}c_{1,3}x_3 \\ (\pi^{Z_{1,2}}x_2+\dots+\pi^k i_k) \\ x_3 \end{pmatrix} \mathrm{d}\mathcal{X} \\ & + f(c_{1,2}i_k)f(\xi c_{1,2}i_k) \int_{\mathfrak{D}_\nu} \int_{\mathfrak{D}_\nu} \int_{\pi^{k-(Z_{1,2}-Z_{1,3})+1}\mathfrak{D}_\nu} \mathrm{d}\mathcal{X} \\ = {} & f(\xi c_{1,2}i_k) \frac{(\rho^{k-(Z_{1,2}-Z_{1,3})} - 1)}{m} \quad + \quad L^{(\xi)}(i_k), \end{aligned}$$

where the first integral in this expression was calculated using the fact that $f$ is fundamental and the third term is trivial since $f.f\xi = 0$.

Therefore it remains only to consider the second integral which we shall refer to as $L^{(\xi)}(i_k)$. Making the appropriate change of variable this becomes,

$$L^{(\xi)}(i_k) = f(\xi c_{1,2} i_k) \int_{\mathfrak{O}_\nu} \int_{\mathfrak{O}_\nu} \int_{\mathfrak{O}_\nu \backslash \pi \mathfrak{O}_\nu} f \begin{pmatrix} \pi^{k-Z_{1,2}}(c_{1,3}x_3 + c_{1,2}i_k) + \ldots + (x_1 + c_{1,2}x_2) \\ \pi^k i_k \\ \pi^{k-(Z_{1,2}-Z_{1,3})} x_3 \end{pmatrix} \mathrm{d}\mathcal{X}$$

$$= f(\xi c_{1,2} i_k) \int_{\mathfrak{O}_\nu} \int_{\mathfrak{O}_\nu} \int_{\mathfrak{O}_\nu \backslash \pi \mathfrak{O}_\nu} f \begin{pmatrix} (c_{1,3}x_3 + c_{1,2}i_k) + \ldots + \pi^{Z_{1,2}-k}(x_1 + c_{1,2}x_2) \\ \pi^{Z_{1,2}} i_k \\ \pi^{Z_{1,3}} x_3 \end{pmatrix} \mathrm{d}\mathcal{X}.$$

If we make the substitution $x_1 \longmapsto x_1 - c_{1,2}x_2$, $\mathrm{d}\mathcal{X} \longmapsto d\mathcal{X}$ and then split the integral into two parts we find,

$$\begin{aligned} L^{(\xi)}(i_k) &= f(\xi c_{1,2} i_k) \underset{c_{1,3}x_3 + c_{1,2}i_k \not\equiv 0(\pi)}{\int_{\mathfrak{O}_\nu} \int_{\mathfrak{O}_\nu} \int_{\mathfrak{O}_\nu \backslash \pi \mathfrak{O}_\nu}} f(c_{1,3}x_3 + c_{1,2}i_k) \mathrm{d}\mathcal{X} \\ &\quad + f(\xi c_{1,2} i_k) \underset{c_{1,3}x_3 + c_{1,2}i_k \equiv 0(\pi)}{\int_{\mathfrak{O}_\nu} \int_{\mathfrak{O}_\nu} \int_{\mathfrak{O}_\nu \backslash \pi \mathfrak{O}_\nu}} f \begin{pmatrix} (c_{1,3}x_3 + c_{1,2}i_k) + \ldots + \pi^{Z_{1,2}-k} x_1 \\ \pi^{Z_{1,2}} i_k \\ \pi^{Z_{1,3}} x_3 \end{pmatrix} \mathrm{d}\mathcal{X} \\ &=: L_1^{(\xi)}(i_k) + L_2^{(\xi)}(i_k), \end{aligned}$$

where, by Lemma 5.5.2, the first integral in this expression is found to satisfy

$$\sum_{i_k \in \Pi \backslash 0} L_1^{(\xi)}(i_k) = \frac{(\rho - 1)}{m}.$$

In order to calculate the second integral we must first make the substitution,

$$x_3 \longmapsto -c_{1,2}c_{1,3}^{-1} i_k + \pi x_3, \quad \mathrm{d}\mathcal{X} \longmapsto \mathrm{d}\mathcal{X} \qquad (\mathrm{mod}\ m),$$

and then use the change of variables,

$$\begin{aligned} x_3 \longmapsto x_3 - c_{1,2}c_{1,3}^{-1} i_{k+1} - \ldots - \pi^{Z_{1,2}-k-1} c_{1,3}^{-1} x_1, &\qquad x_1 \longmapsto x_1, \\ \Rightarrow \quad \mathrm{d}\mathcal{X} \longmapsto \mathrm{d}\mathcal{X} \qquad (\mathrm{mod}\ m).& \end{aligned}$$

Carrying out each of these in turn we find,

$$\begin{aligned} L_2^{(\xi)}(i_k) &= f(\xi c_{1,2} i_k) \int_{\mathfrak{O}_\nu^3} f \begin{pmatrix} \pi(c_{1,3}x_3 + c_{1,2}i_{k+1}) + \ldots + \pi^{Z_{1,2}-k} x_1 \\ \pi^{Z_{1,2}} i_k \\ \pi^{Z_{1,3}}(-c_{1,2}c_{1,3}^{-1} i_k) \end{pmatrix} \mathrm{d}\mathcal{X} \\ &= f(\xi c_{1,2} i_k) \int_{\mathfrak{O}_\nu^3} f \begin{pmatrix} \pi c_{1,3} x_3 \\ \pi^{Z_{1,2}} i_k \\ -\pi^{Z_{1,3}} c_{1,2} c_{1,3}^{-1} i_k \end{pmatrix} \mathrm{d}\mathcal{X}. \end{aligned} \tag{$\spadesuit$}$$

In order to solve this integral we shall now have to further split this into two cases depending on which of the exponents $Z_{1,2}$ and $Z_{1,3}$ is smallest.

Let us suppose that $Z_{1,2} \le Z_{1,3}$. Then the integral $L_2^{(\xi)}(i_k)$ satisfies,

$$\begin{aligned}
L_2^{(\xi)}(i_k) &= f(\xi c_{1,2} i_k) \int_{\mathfrak{O}_\nu^3} f\left(\tfrac{\pi c_{1,3} x_3}{\pi^{Z_{1,2}} i_k}\right) \mathrm{d}\mathcal{X} \\
&= f(\xi c_{1,2} i_k) \int_{\mathfrak{O}_\nu \setminus \pi^{Z_{1,2}} \mathfrak{O}_\nu} f(c_{1,3} x_3) \mathrm{d}\mathcal{X} + f(i_k) f(\xi c_{1,2} i_k) \int_{\pi^{Z_{1,2}} \mathfrak{O}_\nu} \mathrm{d}\mathcal{X} \\
&= f(\xi c_{1,2} i_k) \frac{(\rho^{Z_{1,2}} - 1)}{m} + f(i_k) f(\xi c_{1,2} i_k).
\end{aligned}$$

Now let us suppose that $Z_{1,3} < Z_{1,2}$. Then the integral $L_2^{(\xi)}(i_k)$ satisfies,

$$\begin{aligned}
L_2^{(\xi)}(i_k) &= f(\xi c_{1,2} i_k) \int_{\mathfrak{O}_\nu^3} f\left(\tfrac{\pi c_{1,3} x_3}{-\pi^{Z_{1,3}} c_{1,2} c_{1,3}^{-1} i_k}\right) \mathrm{d}\mathcal{X} \\
&= f(\xi c_{1,2} i_k) \int_{\mathfrak{O}_\nu \setminus \pi^{Z_{1,3}} \mathfrak{O}_\nu} f(c_{1,3} x_3) \mathrm{d}\mathcal{X} + f(-c_{1,2} c_{1,3}^{-1} i_k) f(\xi c_{1,2} i_k) \int_{\pi^{Z_{1,3}} \mathfrak{O}_\nu} \mathrm{d}\mathcal{X} \\
&= f(\xi c_{1,2} i_k) \frac{(\rho^{Z_{1,3}} - 1)}{m} + f(-c_{1,2} c_{1,3}^{-1} i_k) f(\xi c_{1,2} i_k).
\end{aligned}$$

Finally, putting all of our results for the integral $I_2^{(\xi)}(i_k)$ together we have found,

$$I_2^{(\xi)}(i_k) \; = \; f(\xi c_{1,2} i_k) \frac{(\rho^{k-(Z_{1,2}-Z_{1,3})} - 1)}{m} + L_1^{(\xi)}(i_k) + L_2^{(\xi)}(i_k).$$

Therefore, whenever $Z_{1,2} - Z_{1,3} \le k$ the integral $I_2^{(\xi)}(i_k)$ satisfies,

$Z_{1,2} \le Z_{1,3}$ :

$$\begin{aligned}
\prod_{\xi \in \mu_m} \xi^{\sum_{i_k} - I_2^{(\xi)}(i_k)} &= (-1)^{\frac{(\rho-1)}{m}(k-(Z_{1,2}-Z_{1,3}))} . (-1)^{\frac{(\rho-1)}{m}} . (-1)^{\frac{(\rho-1)}{m} Z_{1,2}} . (c_{1,2}, \pi)_{\nu,m} \\
&= (c_{1,2}, \pi)_{\nu,m} . (-1)^{\frac{(\rho-1)}{2^r}(k+1)} . (-1)^{\frac{(\rho-1)}{2^r} Z_{1,3}}
\end{aligned}$$

$Z_{1,3} < Z_{1,2}$ :

$$\begin{aligned}
\prod_{\xi \in \mu_m} \xi^{\sum_{i_k} - I_2^{(\xi)}(i_k)} &= (-1)^{\frac{(\rho-1)}{m}(k-(Z_{1,2}-Z_{1,3}))} . (-1)^{\frac{(\rho-1)}{m}} . (-1)^{\frac{(\rho-1)}{m} Z_{1,3}} . (-c_{1,3}, \pi)_{\nu,m} \\
&= (c_{1,3}, \pi)_{\nu,m} . (-1)^{\frac{(\rho-1)}{2^r} k} . (-1)^{\frac{(\rho-1)}{2^r} Z_{1,2}}.
\end{aligned}$$

Considering the results given in sections $I_1^{(\xi)}(i_k)$ and $I_2^{(\xi)}(i_k)$, when calculating

$$dec_\nu(n_{1,3}, n_{1,2}) = \prod_{k=0}^{Z_{1,2}-1} \prod_{\xi \in \mu_m} \xi^{\sum_{i_k} I_1^{(\xi)}(i_k)} \prod_{k=0}^{Z_{1,2}-1} \prod_{\xi \in \mu_m} \xi^{-\sum_{i_k} I_2^{(\xi)}(i_k)},$$

it shall be sufficient for us to consider just three distinct cases.

**Results for** $0 < Z_{1,3} < Z_{1,2}$ :

$$dec_\nu(n_{1,3}, n_{1,2}) = \prod_{k=0}^{Z_{1,3}-1} \prod_{\xi\in\mu_m} \xi^{\sum_{i_k} I_1^{(\xi)}(i_k)} \prod_{k=Z_{1,2}-Z_{1,3}}^{Z_{1,2}-1} \prod_{\xi\in\mu_m} \xi^{-\sum_{i_k} I_2^{(\xi)}(i_k)}$$
$$= (-1)^{\frac{(\rho-1)}{2^r} Z_{1,3}} (-1)^{\frac{(\rho-1)}{2^r} \frac{Z_{1,3}(Z_{1,3}-1)}{2}} .(-1)^{\frac{(\rho-1)}{2^r} \frac{Z_{1,3}(Z_{1,3}+1)}{2}} (c_{1,3}, \pi)_{\nu,m}^{Z_{1,3}}$$
$$= (c_{1,3}, \pi)_{\nu,m}^{Z_{1,3}}$$

**Results for** $0 < Z_{1,2} \leq Z_{1,3}$ :

$$dec_\nu(n_{1,3}, n_{1,2}) = (-1)^{\frac{(\rho-1)}{2^r} Z_{1,2}Z_{1,3}} (-1)^{\frac{(\rho-1)}{2^r} \frac{Z_{1,2}(Z_{1,2}-1)}{2}}$$
$$.(-1)^{\frac{(\rho-1)}{2^r} Z_{1,2}Z_{1,3}} (-1)^{\frac{(\rho-1)}{2^r} \frac{Z_{1,2}(Z_{1,2}+1)}{2}} (c_{1,2}, \pi)_{\nu,m}^{Z_{1,2}}$$
$$= (-1)^{\frac{(\rho-1)}{2^r} Z_{1,2}} (c_{1,2}, \pi)_{\nu,m}^{Z_{1,2}}$$

**Results for** $0 < Z_{1,2} \leq Z_{1,3}$ :

$$dec_\nu(n_{1,3}, n_{1,2}) = \prod_{k=0}^{Z_{1,2}-1} \prod_{\xi\in\mu_m} \xi^{\sum_{i_k} I_1^{(\xi)}(i_k)} \prod_{k=0}^{Z_{1,2}-1} \prod_{\xi\in\mu_m} \xi^{-\sum_{i_k} I_2^{(\xi)}(i_k)} = 1.$$

As we had hoped, these results are indeed consistent with those stated in the theorem and our proof is complete.

□

**Theorem 6.4.3** *For each,*

$$n_{2,3} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \pi^{-Z_{2,3}} c_{2,3} \\ 0 & 0 & 1 \end{pmatrix}, \qquad n_{1,2} := \begin{pmatrix} 1 & \pi^{-Z_{1,2}} c_{1,2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in N,$$

*the cocycle $dec_\nu$ satisfies,*

- $Z_{1,2}, Z_{2,3} \leq 0$ :
$$dec_\nu(n_{2,3}, n_{1,2}) = 1$$
- $0 < Z_{1,2} < Z_{2,3}$ :
$$dec_\nu(n_{2,3}, n_{1,2}) = (c_{2,3}, \pi)_{\nu,m}^{-Z_{1,2}} (-1)^{\frac{(\rho-1)}{2^r} \frac{Z_{1,2}(Z_{1,2}+1)}{2}}$$
- $0 < Z_{2,3} \leq Z_{1,2}$ :
$$dec_\nu(n_{2,3}, n_{1,2}) = (c_{2,3}, \pi)_{\nu,m}^{-Z_{1,2}} (-1)^{\frac{(\rho-1)}{2^r} Z_{1,2}Z_{2,3}} (-1)^{\frac{(\rho-1)}{2^r} \frac{Z_{2,3}(Z_{2,3}-1)}{2}}.$$

**PROOF OF THEOREM:**

Once again, using Theorems 6.2.1 and 6.2.2, whenever $Z_{2,3} < 0$ or $Z_{1,2} \le 0$ we are able to deduce that,

$$dec_\nu(n_{2,3}, n_{1,2}) \;=\; 1.$$

Therefore, for the remainder of this proof we may assume $Z_{2,3} \ge 0$ and $Z_{1,2} > 0$.

### The Integrals $I_1^{(\xi)}(i_k)$

Since $Z_{1,2} > 0$ we may again use the dissection given in Section 6.2.1. Having made the substitution,

$$x_2 \longmapsto \pi^{Z_{1,2}}x_2 + \pi^{Z_{1,2}-1}i_{Z_{1,2}-1} + \ldots + \pi^k i_k \qquad \Rightarrow \mathrm{d}x_2 \longmapsto |\pi^{Z_{1,2}}|_\nu \mathrm{d}x_2 \equiv \mathrm{d}x_2 \pmod m,$$

we are able to write,

$$I_1^{(\xi)}(i_k) = \int_{\mathfrak{O}_\nu^3} f\left(\begin{smallmatrix} x_1 \\ (\pi^{Z_{1,2}}x_2+\ldots+\pi^k i_k)+\pi^{-Z_{2,3}}c_{2,3}x_3 \\ x_3 \end{smallmatrix}\right) f\left(\begin{smallmatrix} \xi x_1 \\ \xi(\cdot+\pi^k i_k) \\ \xi x_3 \end{smallmatrix}\right) \mathrm{d}\mathcal{X}.$$

Since we are assuming that $Z_{2,3} \ge 0$ for this integral we need only consider one case. To begin we shall split the integral with respect to the third variable $x_3$ as follows:

$$\begin{aligned} I_1^{(\xi)}(i_k) = &\int_{\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu \setminus \pi^{k+Z_{2,3}}\mathfrak{O}_\nu} f\left(\begin{smallmatrix} x_1 \\ \pi^{-Z_{2,3}}c_{2,3}x_3 \end{smallmatrix}\right) f\left(\begin{smallmatrix} \xi x_1 \\ \xi\pi^k i_k \\ \xi x_3 \end{smallmatrix}\right) \mathrm{d}\mathcal{X} \\ &+ \int_{\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu}\int_{\pi^{k+Z_{2,3}}\mathfrak{O}_\nu \setminus \pi^{k+Z_{2,3}+1}\mathfrak{O}_\nu} f\left(\begin{smallmatrix} x_1 \\ (\pi^{Z_{1,2}}x_2+\ldots+\pi^k i_k)+\pi^{-Z_{2,3}}c_{2,3}x_3 \\ x_3 \end{smallmatrix}\right) f\left(\begin{smallmatrix} \xi x_1 \\ \xi\pi^k i_k \end{smallmatrix}\right) \mathrm{d}\mathcal{X} \\ &+ \int_{\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu}\int_{\pi^{k+Z_{2,3}+1}\mathfrak{O}_\nu} f\left(\begin{smallmatrix} x_1 \\ \pi^k i_k \end{smallmatrix}\right) f\left(\begin{smallmatrix} \xi x_1 \\ \xi\pi^k i_k \end{smallmatrix}\right) \mathrm{d}\mathcal{X}. \end{aligned}$$

Since the last term is trivial, as $f.f\xi = 0$, we need only consider the first two integrals.

For the first term we split the space over which we are integrating and keep only the part which depends on $i_k$. For the second integral we make the change of variable $x_3 \mapsto \pi^{k+Z_{2,3}}x_3$. Having done this the integral $I_1^{(\xi)}(i_k)$ becomes,

$$\begin{aligned} I_1^{(\xi)}(i_k) &= f(\xi i_k)\int_{\pi^{k+1}\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu}\int_{\pi^k\mathfrak{O}_\nu \setminus \pi^{k+Z_{2,3}}\mathfrak{O}_\nu} f\left(\begin{smallmatrix} x_1 \\ \pi^{-Z_{2,3}}c_{2,3}x_3 \end{smallmatrix}\right) \mathrm{d}\mathcal{X} \\ &\qquad + \int_{\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu \setminus \pi\mathfrak{O}_\nu} f\left(\begin{smallmatrix} x_1 \\ \pi^k(c_{2,3}x_3+i_k)+\ldots+\pi^{Z_{1,2}}x_2 \\ \pi^{k+Z_{2,3}}x_3 \end{smallmatrix}\right) f\left(\begin{smallmatrix} \xi x_1 \\ \xi\pi^k i_k \end{smallmatrix}\right) \mathrm{d}\mathcal{X} \\ &= f(\xi i_k)\frac{(\rho^{Z_{2,3}}-1)}{m} \;+\; L^{(\xi)}(i_k), \end{aligned}$$

where the first integral was calculated using the fact that $f$ is fundamental and we have defined the second integral to be $L^{(\xi)}(i_k)$.

By once again splitting the space over which we are integrating and keeping only the term which depends on $i_k$ we find that $L^{(\xi)}(i_k)$ satisfies,

$$L^{(\xi)}(i_k) = \int_{\pi^{k+1}\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu\setminus\pi\mathfrak{O}_\nu} f\left(\begin{smallmatrix} x_1 \\ \pi^k(c_{2,3}x_3+i_k)+\dots \\ \pi^{k+Z_{2,3}}x_3 \end{smallmatrix}\right) f(\xi i_k)\mathrm{d}\mathcal{X},$$

substituting $x_1 \mapsto \pi^{k+1}x_1$ this becomes,

$$= \int_{\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu\setminus\pi\mathfrak{O}_\nu} f\left(\begin{smallmatrix} \pi x_1 \\ (c_{2,3}x_3+i_k)+\dots+\pi^{Z_{1,2}-k}x_2 \\ \pi^{Z_{2,3}}x_3 \end{smallmatrix}\right) f(\xi i_k)\mathrm{d}\mathcal{X} \qquad (\Diamond)$$

$$= f(\xi i_k)\int_{\substack{\mathfrak{O}_\nu \\ (c_{2,3}x_3+i_k)\not\equiv 0(\pi)}}\int_{\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu\setminus\pi\mathfrak{O}_\nu} f(c_{2,3}x_3+i_k)\mathrm{d}\mathcal{X}$$

$$+ f(\xi i_k)\int_{\substack{\mathfrak{O}_\nu \\ (c_{2,3}x_3+i_k)\equiv 0(\pi)}}\int_{\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu\setminus\pi\mathfrak{O}_\nu} f\left(\begin{smallmatrix} \pi x_1 \\ (c_{2,3}x_3+i_k)+\dots+\pi^{Z_{1,2}-k}x_2 \\ \pi^{Z_{2,3}}x_3 \end{smallmatrix}\right)\mathrm{d}\mathcal{X}$$

$$=: L_1^{(\xi)}(i_k) + L_2^{(\xi)}(i_k), \quad \text{defined respectively.}$$

Using Lemma 5.5.2 the integral $L_1^{(\xi)}(i_k)$ is found to satisfy,

$$\sum_{i_k\in\Pi\setminus 0} L_1^{(\xi)}(i_k) = \frac{(\rho-1)}{m}.$$

In order to calculate $L_2^{(\xi)}(i_k)$ we must first make the substitution,

$$x_3 \longmapsto -c_{2,3}^{-1}i_k + \pi x_3, \quad \mathrm{d}\mathcal{X} \longmapsto \mathrm{d}\mathcal{X} \qquad (\mathrm{mod}\ m),$$

and then use the change of variables,

$$x_3 \mapsto x_3 - c_{2,3}^{-1}i_{k+1} - \dots - \pi^{Z_{1,2}-k-1}c_{2,3}^{-1}x_2, \qquad x_2 \mapsto x_2, \qquad \mathrm{d}\mathcal{X} \mapsto \mathrm{d}\mathcal{X} \qquad (\mathrm{mod}\ m).$$

Carrying out each of these in turn we find,

$$L_2^{(\xi)}(i_k) = f(\xi i_k)\int_{\mathfrak{O}_\nu^3} f\left(\begin{smallmatrix} \pi x_1 \\ \pi(c_{2,3}x_3+i_{k+1})+\dots+\pi^{Z_{1,2}-k}x_2 \\ -\pi^{Z_{2,3}}c_{2,3}^{-1}i_k \end{smallmatrix}\right)\mathrm{d}\mathcal{X}$$

$$= f(\xi i_k)\int_{\mathfrak{O}_\nu^3} f\left(\begin{smallmatrix} x_1 \\ c_{2,3}x_3 \\ -\pi^{Z_{2,3}-1}c_{2,3}^{-1}i_k \end{smallmatrix}\right)\mathrm{d}\mathcal{X}$$

$$= f(-c_{2,3}^{-1}i_k)f(\xi i_k) \qquad \text{by Lemma 6.3.1.}$$

Finally, by assembling all of our results we are able to conclude that the integrals $I_1^{(\xi)}(i_k)$ satisfy,

$$\sum_{i_k \in \Pi\backslash 0} I_1^{(\xi)}(i_k) = \frac{(\rho-1)}{m}(Z_{2,3}) + \frac{(\rho-1)}{m} + \sum_{i_k} f(-c_{2,3}^{-1} i_k) f(\xi i_k)$$

and therefore,

$$\begin{aligned}\prod_{\xi \in \mu_m} \xi^{\sum_{i_k} I_1^{(\xi)}(i_k)} &= (-1)^{\frac{(\rho-1)}{2^r} Z_{2,3}} (-1)^{\frac{(\rho-1)}{2^r}} (-c_{2,3}^{-1}, \pi)_{\nu,m} \\ &= (-1)^{\frac{(\rho-1)}{2^r} Z_{2,3}} (c_{2,3}^{-1}, \pi)_{\nu,m}.\end{aligned}$$

### The Integrals $I_2^{(\xi)}(i_k)$

To solve these integrals we use the substitutions,

$$x_1 \longmapsto x_1 + \pi^{-Z_{1,2}} c_{1,2}(\pi^{Z_{1,2}} x_2 + \pi^{Z_{1,2}-1} i_{Z_{1,2}-1} + \ldots + \pi^k i_k) \qquad x_2 \longmapsto \pi^{Z_{1,2}} x_2 + \ldots + \pi^k i_k$$
$$\Rightarrow \mathrm{d}\mathcal{X} \longmapsto \rho^{-Z_{1,2}} \mathrm{d}\mathcal{X} \equiv \mathrm{d}\mathcal{X} \pmod{m},$$

which allow us to write,

$$\begin{aligned}I_2^{(\xi)}(i_k) &= \int_{\mathfrak{O}_\nu^3} f\begin{pmatrix} x_1 + \pi^{-Z_{1,2}} c_{1,2}(\ldots + \pi^k i_k) \\ (\pi^{Z_{1,2}} x_2 + \ldots + \pi^k i_k) + \pi^{-Z_{2,3}} c_{2,3} x_3 \\ x_3 \end{pmatrix} f(\xi c_{1,2} i_k) \mathrm{d}\mathcal{X} \\ &= f(\xi c_{1,2} i_k) \int_{\mathfrak{O}_\nu^3} f\begin{pmatrix} \pi^{k-Z_{1,2}} c_{1,2} i_k \\ \pi^{-Z_{2,3}} c_{2,3} x_3 \\ x_3 \end{pmatrix} \mathrm{d}\mathcal{X}.\end{aligned}$$

This integral will split into two cases depending on the value of $k$. For each of these we calculate,

$k \leq Z_{1,2} - Z_{2,3}$ :

$$I_2^{(\xi)}(i_k) = f(c_{1,2} i_k) f(\xi c_{1,2} i_k) = 0,$$

$k > Z_{1,2} - Z_{2,3}$ :

$$\begin{aligned}I_2^{(\xi)}(i_k) &= f(\xi c_{1,2} i_k) \int_{\mathfrak{O}_\nu} \int_{\mathfrak{O}_\nu} \int_{\mathfrak{O}_\nu \backslash \pi^{k-(Z_{1,2}-Z_{2,3})} \mathfrak{O}_\nu} f(c_{2,3} x_3) \mathrm{d}\mathcal{X} \\ &\quad + f(c_{1,2} i_k) f(\xi c_{1,2} i_k) \int_{\mathfrak{O}_\nu} \int_{\mathfrak{O}_\nu} \int_{\pi^{k-(Z_{1,2}-Z_{2,3})} \mathfrak{O}_\nu} \mathrm{d}\mathcal{X} \\ &= f(\xi c_{1,2} i_k) \frac{(\rho^{k-(Z_{1,2}-Z_{2,3})} - 1)}{m}.\end{aligned}$$

Finally we are able to conclude that the integrals $I_2^{(\xi)}(i_k)$ satisfy,

$k \leq Z_{1,2} - Z_{2,3}$ :

$$\prod_{\xi \in \mu_m} \xi^{\sum_{i_k} -I_2^{(\xi)}(i_k)} = 1$$

$k > Z_{1,2} - Z_{2,3}$ :

$$\prod_{\xi \in \mu_m} \xi^{\sum_{i_k} -I_2^{(\xi)}(i_k)} = (-1)^{\frac{(\rho-1)}{2^r}(Z_{1,2}+Z_{2,3})}(-1)^{\frac{(\rho-1)}{2^r}k}.$$

When we consider the possible results for the cocycle $dec_\nu$ we find that we shall again have three distinct cases.

**Results for** $0 < Z_{1,2} < Z_{2,3}$ :

$$\begin{aligned} dec_\nu(n_{2,3}, n_{1,2}) &= \prod_{k=0}^{Z_{1,2}-1} \prod_{\xi \in \mu_m} \xi^{\sum_{i_k} I_1^{(\xi)}(i_k)} \prod_{k=0}^{Z_{1,2}-1} \prod_{\xi \in \mu_m} \xi^{-\sum_{i_k} I_2^{(\xi)}(i_k)} \\ &= (-1)^{\frac{(\rho-1)}{m} Z_{1,2}Z_{2,3}} (c_{2,3}^{-1}, \pi)_{\nu,m}^{Z_{1,2}} . (-1)^{\frac{(\rho-1)}{m} Z_{1,2}(Z_{1,2}+Z_{2,3})} (-1)^{\frac{(\rho-1)}{m} \frac{Z_{1,2}(Z_{1,2}-1)}{2}} \\ &= (c_{2,3}, \pi)_{\nu,m}^{-Z_{1,2}} (-1)^{\frac{(\rho-1)}{2^r} \frac{Z_{1,2}(Z_{1,2}+1)}{2}}. \end{aligned}$$

**Results for** $0 < Z_{2,3} \leq Z_{1,2}$ :

$$\begin{aligned} dec_\nu(n_{2,3}, n_{1,2}) &= \prod_{k=0}^{Z_{1,2}-1} \prod_{\xi \in \mu_m} \xi^{\sum_{i_k} I_1^{(\xi)}(i_k)} \prod_{k=0}^{Z_{1,2}-1} \prod_{\xi \in \mu_m} \xi^{-\sum_{i_k} I_2^{(\xi)}(i_k)} \\ &= \prod_{k=0}^{Z_{1,2}-1} \prod_{\xi \in \mu_m} \xi^{\sum_{i_k} I_1^{(\xi)}(i_k)} \prod_{k=Z_{1,2}-Z_{2,3}+1}^{Z_{1,2}-1} \prod_{\xi \in \mu_m} \xi^{-\sum_{i_k} I_2^{(\xi)}(i_k)} \\ &= (-1)^{\frac{(\rho-1)}{m} Z_{1,2}Z_{2,3}} (c_{2,3}^{-1}, \pi)_{\nu,m}^{Z_{1,2}} . (-1)^{\frac{(\rho-1)}{m} \frac{Z_{2,3}(Z_{2,3}-1)}{2}} \\ &= (c_{2,3}, \pi)_{\nu,m}^{-Z_{1,2}} (-1)^{\frac{(\rho-1)}{2^r} Z_{1,2}Z_{2,3}} (-1)^{\frac{(\rho-1)}{2^r} \frac{Z_{2,3}(Z_{2,3}-1)}{2}}. \end{aligned}$$

**Results for** $0 = Z_{2,3} < Z_{1,2}$ :

$$dec_\nu(n_{2,3}, n_{1,2}) = \prod_{k=0}^{Z_{1,2}-1} \prod_{\xi \in \mu_m} \xi^{\sum_{i_k} I_1^{(\xi)}(i_k)} \prod_{k=0}^{Z_{1,2}-1} \prod_{\xi \in \mu_m} \xi^{-\sum_{i_k} I_2^{(\xi)}(i_k)} = 1.$$

Which completes the proof of our theorem. □

**Theorem 6.4.4** *Let us define the matrices $n_{i,j} \in N$ by,*

$$n_{i,j} = n_{i,j}(\pi^{-Z_{i,j}} c_{i,j}) \in N, \quad \text{for each } 1 \le i < j \le 3.$$

*Then we find that the cocycle $dec_\nu$ satisfies,*

$\bullet Z_{1,2} \le 0:$ $\quad dec_\nu(n_{2,3}n_{1,3}, n_{1,2}) = 1$

$\bullet Z_{2,3} < 0:$ $\quad dec_\nu(n_{2,3}n_{1,3}, n_{1,2}) = dec_\nu(n_{1,3}, n_{1,2})$ *results on page 159*

$\bullet Z_{1,3} < 0:$ $\quad dec_\nu(n_{2,3}n_{1,3}, n_{1,2}) = dec_\nu(n_{2,3}, n_{1,2})$ *results on page 166.*

*Now let us suppose that the exponents satisfy $Z_{1,3}, Z_{2,3} \ge 0$ and $Z_{1,2} > 0$. If we define,*

$$(1 - c_{2,3}c_{1,2}c_{1,3}^{-1}) = \pi^E e,$$

*for some $E \in \mathbb{Z}^{\ge 0}$ and where either $e = 0$ or $|\, e\,|_\nu = 1$, we find that the cocycle $dec_\nu$ satisfies,*

$\bullet\ Z_{1,3} - Z_{2,3} < 0 < Z_{2,3} \le Z_{1,2}:$

$$dec_\nu(n_{2,3}n_{1,3}, n_{1,2}) = (c_{2,3}, \pi)_{\nu,m}^{-Z_{1,2}} (-1)^{\frac{(\rho-1)}{2^r} Z_{2,3}Z_{1,2}} (-1)^{\frac{(\rho-1)}{2^r} \frac{Z_{2,3}(Z_{2,3}-1)}{2}}$$

$\bullet\ Z_{1,3} - Z_{2,3} < 0 < Z_{1,2} < Z_{2,3}:$

$$dec_\nu(n_{2,3}n_{1,3}, n_{1,2}) = (c_{2,3}, \pi)_{\nu,m}^{-Z_{1,2}} (-1)^{\frac{(\rho-1)}{2^r} \frac{Z_{1,2}(Z_{1,2}+1)}{2}}$$

$\bullet\ 0 \le Z_{1,3} - Z_{2,3} \le Z_{1,3} < Z_{1,2}:$

$$dec_\nu(n_{2,3}n_{1,3}, n_{1,2}) = (c_{2,3}, \pi)_{\nu,m}^{-(Z_{1,2}+Z_{2,3})} (c_{1,3}, \pi)_{\nu,m}^{Z_{1,3}} (-1)^{\frac{(\rho-1)}{2^r} Z_{2,3}Z_{1,2}} (-1)^{\frac{(\rho-1)}{2^r} \frac{Z_{2,3}(Z_{2,3}-1)}{2}}$$

$\bullet\ 0 < Z_{1,2} < Z_{1,3} - Z_{2,3} \le Z_{1,3}:$

$$dec_\nu(n_{2,3}n_{1,3}, n_{1,2}) = (c_{1,2}, \pi)_{\nu,m}^{Z_{1,2}} (-1)^{\frac{(\rho-1)}{2^r} Z_{1,2}}$$

$\bullet\ 0 \le Z_{1,3} - Z_{2,3} < Z_{1,2} \le Z_{1,3}:$

$$dec_\nu(n_{2,3}n_{1,3}, n_{1,2}) = (c_{2,3}, \pi)_{\nu,m}^{(Z_{1,3}-Z_{2,3})-Z_{1,2}} (c_{1,3}/c_{2,3}, \pi)_{\nu,m}^{Z_{1,2}} (-1)^{\frac{(\rho-1)}{2^r} Z_{2,3}Z_{1,3}}$$
$$(-1)^{\frac{(\rho-1)}{2^r} \frac{Z_{1,2}(Z_{1,2}+1)}{2}} (-1)^{\frac{(\rho-1)}{2^r} \frac{Z_{1,3}(Z_{1,3}+1)}{2}} (-1)^{\frac{(\rho-1)}{2^r} \frac{Z_{2,3}(Z_{2,3}-1)}{2}}$$

$\bullet\ 0 < Z_{1,2} = Z_{1,3} - Z_{2,3} \le Z_{1,3}:$

$\circ\ E \le Z_{2,3}:$

$$dec_\nu(n_{2,3}n_{1,3}, n_{1,2}) = (c_{1,2}, \pi)_{\nu,m}^{Z_{1,2}} (e, \pi)_{\nu,m}^{-Z_{1,2}} (-1)^{\frac{(\rho-1)}{2^r} Z_{1,2}} (-1)^{\frac{(\rho-1)}{2^r} Z_{1,2}E}$$

$\circ\ E > Z_{2,3}$ or $e = 0:$

$$dec_\nu(n_{2,3}n_{1,3}, n_{1,2}) = (c_{1,3}, \pi)_{\nu,m}^{Z_{1,2}} (-1)^{\frac{(\rho-1)}{2^r} Z_{1,3}} (-1)^{\frac{(\rho-1)}{2^r} Z_{1,3}Z_{1,2}}.$$

**PROOF OF THEOREM:**

Using Theorems 6.2.1 and 6.2.2 the first three results quickly follow. Therefore, for the remainder of this proof we shall assume that $Z_{1,3},\ Z_{2,3} \geq 0$ and $Z_{1,2} > 0$ and calculate the integrals $I_1^{(\xi)}(i_k)$ and $I_2^{(\xi)}(i_k)$.

### The Integrals $I_1^{(\xi)}(i_k)$

Having employed the substitutions relating to the dissection with respect to $n_{1,2}$ we are able to write,

$$I_1^{(\xi)}(i_k) = \int_{\mathfrak{D}_\nu^3} f\begin{pmatrix} x_1+\pi^{-Z_{1,3}}c_{1,3}x_3 \\ (\pi^{Z_{1,2}}x_2+.+\pi^k i_k)+\pi^{-Z_{2,3}}c_{2,3}x_3 \\ x_3 \end{pmatrix} f\begin{pmatrix} \xi x_1 \\ \xi(.+\pi^k i_k) \\ \xi x_3 \end{pmatrix} \mathrm{d}\mathcal{X}.$$

In order to calculate this integral we shall once again need to split this into many different cases depending on the value of the exponents $Z_{i,j}$ and the integer $k$. By doing this we shall also be able to use some of the previous work from this chapter.

### Case (1) $\quad Z_{1,3} < Z_{2,3}$ :

To begin with we shall consider the case when $Z_{1,3} < Z_{2,3}$. Now, although it is not strictly true, in this case we shall almost have

$$f\begin{pmatrix} x_1+\pi^{-Z_{1,3}}c_{1,3}x_3 \\ (\pi^{Z_{1,2}}x_2+.+\pi^k i_k)+\pi^{-Z_{2,3}}c_{2,3}x_3 \\ x_3 \end{pmatrix} \approx f\begin{pmatrix} x_1 \\ (\pi^{Z_{1,2}}x_2+.+\pi^k i_k)+\pi^{-Z_{2,3}}c_{2,3}x_3 \\ x_3 \end{pmatrix}.$$

However, if this statement were true we would have found that this integral $I_1^{(\xi)}(i_k)$ was identical to the integral $I_1^{(\xi)}(i_k)$ given in the calculation of $dec_\nu(n_{2,3}, n_{1,2})$ on page 167. This gives us the idea to go back to that calculation and see what changes the addition of the term $(\pi^{-Z_{1,3}}c_{1,3}x_3)$ makes. Then, although these calculations are not identical they should still be very similar.

So, having replaced $x_1$ with $x_1 + \pi^{-Z_{1,3}}c_{1,3}x_3$ we find that when following through the calculation on page 167, by simply employing the substitution

$$x_1 + \pi^{(Z_{2,3}-Z_{1,3})-1}c_{1,3}x_3 \longmapsto x_1$$

at the point "$(\diamondsuit)$" in that calculation, we are in fact able to eliminate the difference.

Therefore, in the case that $Z_{1,3} < Z_{2,3}$ we do indeed find that the integral $I_1^{(\xi)}(i_k)$ is the same as that given in the calculation of $dec_\nu(n_{2,3}, n_{1,2})$ on page 167. That is,

$$\sum_{i_k \in \Pi \backslash 0} I_1^{(\xi)}(i_k) = \frac{(\rho-1)}{m}(Z_{2,3}) + \frac{(\rho-1)}{m} + \sum_{i_k \in \Pi \backslash 0} f(-c_{2,3}^{-1} i_k) f(\xi i_k)$$

and therefore,

$$\begin{aligned}\prod_{\xi \in \mu_m} \xi^{\sum_{i_k} I_1^{(\xi)}(i_k)} &= (-1)^{\frac{(\rho-1)}{m} Z_{2,3}} (-1)^{\frac{(\rho-1)}{m}} (-c_{2,3}^{-1}, \pi)_{\nu,m} \\ &= (c_{2,3}^{-1}, \pi)_{\nu,m}\, (-1)^{\frac{(\rho-1)}{2^r} Z_{2,3}}.\end{aligned}$$

**Case (2)** $\quad Z_{1,3} \geq Z_{2,3}$ :

Now let us suppose that $Z_{1,3} \geq Z_{2,3}$. In this case we find that the integral $I_1^{(\xi)}(i_k)$ satisfies,

$$\begin{aligned} I_1^{(\xi)}(i_k) &= \int_{\mathfrak{O}_\nu} \int_{\mathfrak{O}_\nu} \int_{\mathfrak{O}_\nu \backslash \pi^{Z_{1,3}} \mathfrak{O}_\nu} f(c_{1,3} x_3) f\begin{pmatrix} \xi x_1 \\ \xi \pi^k i_k \\ \xi x_3 \end{pmatrix} \mathrm{d}\mathcal{X} \\ &\quad + \int_{\mathfrak{O}_\nu^3} f\begin{pmatrix} x_1 + c_{1,3} x_3 \\ (\pi^{Z_{1,2}} x_2 + . + \pi^k i_k) + \pi^{Z_{1,3} - Z_{2,3}} c_{2,3} x_3 \\ \pi^{Z_{1,3}} x_3 \end{pmatrix} f\begin{pmatrix} \xi x_1 \\ \xi \pi^k i_k \\ \xi \pi^{Z_{1,3}} x_3 \end{pmatrix} \mathrm{d}\mathcal{X} \qquad \begin{array}{c}\text{having used} \\ x_3 \longmapsto \pi^{Z_{1,3}} x_3\end{array} \\ &=: L_1^{(\xi)}(i_k) + L_2^{(\xi)}(i_k), \qquad \text{defined respectively.} \end{aligned}$$

### The integral $L_1^{(\xi)}(i_k)$

First of all we shall consider the integral $L_1^{(\xi)}(i_k)$. Referring back to page 160 we see that this integral $L_1^{(\xi)}(i_k)$ is identical to the integral found on pages 160 and 161 in the calculation of $dec_\nu(n_{1,3}, n_{1,2})$. That is,

$$\begin{aligned} k \geq Z_{1,3} : &\qquad L_1^{(\xi)}(i_k) = 0 \quad \text{since this integral has no dependence on } i_k \\ k < Z_{1,3} : &\qquad L_1^{(\xi)}(i_k) = f(\xi i_k) \frac{(\pi^{Z_{1,3}-k} - 1)}{m}. \end{aligned}$$

### The integral $L_2^{(\xi)}(i_k)$

Let us now consider the integral $L_2^{(\xi)}(i_k)$. To begin with we shall split this into two cases depending on which of $k$ and $Z_{1,3} - Z_{2,3} \geq 0$ is greatest.

$L_2^{(\xi)}(i_k)$ **for** $0 \leq k < Z_{1,3} - Z_{2,3}$**:**

Let us suppose that $0 \leq k < Z_{1,3} - Z_{2,3} \leq Z_{1,3}$. Then we have,

$$f\begin{pmatrix} x_1+c_{1,3}x_3 \\ (.+\pi^k i_k)+\pi^{Z_{1,3}-Z_{2,3}}c_{2,3}x_3 \\ \pi^{Z_{1,3}}x_3 \end{pmatrix} = f\begin{pmatrix} x_1+c_{1,3}x_3 \\ (.+\pi^k i_k) \\ \pi^{Z_{1,3}}x_3 \end{pmatrix}.$$

This allows us to deduce that $L_2^{(\xi)}(i_k)$ is identical to the integral given in the calculation of $dec_\nu(n_{1,3}, n_{1,2})$ on page 161. Referring back to that result we once again find that,

$$L_2^{(\xi)}(i_k) = f(i_k)\frac{(\pi^{k+1}-1)}{m} + f(\xi i_k)\frac{(\pi^{k+1}-1)}{m} = 0.$$

$L_2^{(\xi)}(i_k)$ **for** $0 \leq Z_{1,3} - Z_{2,3} \leq k$**:**

Whenever we have $0 \leq Z_{1,3} - Z_{2,3} \leq k$ the integral $L_2^{(\xi)}(i_k)$ satisfies,

$$\begin{aligned} L_2^{(\xi)}(i_k) &= \int_{\mathfrak{O}_\nu \setminus \pi^{k+1}\mathfrak{O}_\nu} \int_{\mathfrak{O}_\nu} \int_{\mathfrak{O}_\nu} f\begin{pmatrix} x_1+c_{1,3}x_3 \\ .+\pi^k i_k+\pi^{Z_{1,3}-Z_{2,3}}c_{2,3}x_3 \\ \pi^{Z_{1,3}}x_3 \end{pmatrix} f\begin{pmatrix} \xi x_1 \\ \xi\pi^{Z_{1,3}}x_3 \end{pmatrix} \mathrm{d}\mathcal{X} \\ &\quad + \int_{\pi^{k+1}\mathfrak{O}_\nu} \int_{\mathfrak{O}_\nu} \int_{\mathfrak{O}_\nu} f\begin{pmatrix} x_1+c_{1,3}x_3 \\ .+\pi^k i_k + \pi^{Z_{1,3}-Z_{2,3}}c_{2,3}x_3 \\ \pi^{Z_{1,3}}x_3 \end{pmatrix} f\begin{pmatrix} \xi\pi^k i_k \\ \xi\pi^{Z_{1,3}}x_3 \end{pmatrix} \mathrm{d}\mathcal{X} \\ &=: L_3^{(\xi)}(i_k) + L_4^{(\xi)}(i_k) \qquad \text{defined respectively.} \end{aligned}$$

Let us begin by considering the integral $L_3^{(\xi)}(i_k)$. By once again splitting this integral and neglecting the term independent of $i_k$ we find,

$$\begin{aligned} L_3^{(\xi)}(i_k) &= \int_{\mathfrak{O}_\nu \setminus \pi^{k+1}\mathfrak{O}_\nu} \int_{\mathfrak{O}_\nu} \int_{\pi^{k-(Z_{1,3}-Z_{2,3})}\mathfrak{O}_\nu \setminus \pi^{k-(Z_{1,3}-Z_{2,3})+1}\mathfrak{O}_\nu} f\begin{pmatrix} x_1+c_{1,3}x_3 \\ .+\pi^k i_k + \pi^{Z_{1,3}-Z_{2,3}}c_{2,3}x_3 \\ \pi^{Z_{1,3}}x_3 \end{pmatrix} f(\xi x_1)\mathrm{d}\mathcal{X} \\ &\quad + \int_{\mathfrak{O}_\nu \setminus \pi^{k+1}\mathfrak{O}_\nu} \int_{\mathfrak{O}_\nu} \int_{\pi^{k-(Z_{1,3}-Z_{2,3})+1}\mathfrak{O}_\nu} f\begin{pmatrix} x_1+c_{1,3}x_3 \\ \pi^k i_k \end{pmatrix} f(\xi x_1)\mathrm{d}\mathcal{X} \\ &=: M_1^{(\xi)}(i_k) + M_2^{(\xi)}(i_k) \qquad \text{defined respectively.} \end{aligned}$$

For the integral $M_2^{(\xi)}(i_k)$, using Lemma 5.2.2 and considering when the integral may depend on $i_k$, we simply find

$$\begin{aligned} M_2^{(\xi)}(i_k) &= \int_{\pi^{k-(Z_{1,3}-Z_{2,3})+1}\mathfrak{O}_\nu \setminus \pi^{k+1}\mathfrak{O}_\nu} \int_{\mathfrak{O}_\nu} \int_{-x_1/c_{1,3}+\pi^{k+1}\mathfrak{O}_\nu} f(i_k)f(\xi x_1)\mathrm{d}\mathcal{X} \\ &= f(i_k)\frac{(\rho^{Z_{1,3}-Z_{2,3}}-1)}{m}. \end{aligned}$$

For the integral $M_1^{(\xi)}(i_k)$ we begin by making the substitution $x_3 \longmapsto \pi^{k-(Z_{1,3}-Z_{2,3})}x_3$. Then, having split the integral with respect to the variable $x_1$ and neglected the part of the integral independent of $i_k$, we have

$$M_1^{(\xi)}(i_k) = \int_{\mathfrak{O}_\nu\setminus\pi\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu\setminus\pi\mathfrak{O}_\nu} f\begin{pmatrix} \pi^{-(Z_{1,3}-Z_{2,3})}(x_1+c_{1,3}x_3) \\ (c_{2,3}x_3+i_k)+\dots \\ \pi^{Z_{2,3}}x_3 \end{pmatrix} f(\xi x_1)\mathrm{d}\mathcal{X}.$$

However, this integral will only be dependent on $i_k$ when $x_1 + c_{1,3}x_3 \in \pi^{(Z_{1,3}-Z_{2,3})+1}\mathfrak{O}_\nu$. Therefore we make the substitution,

$$x_1 \longmapsto -c_{1,3}x_3 + \pi^{(Z_{1,3}-Z_{2,3})+1}x_1$$

after which our integral becomes,

$$\begin{aligned}
M_1^{(\xi)}(i_k) &= \int_{\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu\setminus\pi\mathfrak{O}_\nu} f\begin{pmatrix} \pi x_1 \\ (c_{2,3}x_3+i_k)+\dots \\ \pi^{Z_{2,3}}x_3 \end{pmatrix} f(\xi(-c_{1,3}x_3))\mathrm{d}\mathcal{X} \\
&= \underset{(c_{2,3}x_3+i_k)\not\equiv 0(\pi)}{\int_{\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu\setminus\pi\mathfrak{O}_\nu}} f(c_{2,3}x_3+i_k)f(-\xi c_{1,3}x_3)\mathrm{d}\mathcal{X} \\
&\quad + \underset{(c_{2,3}x_3+i_k)\equiv 0(\pi)}{\int_{\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu}\int_{\mathfrak{O}_\nu\setminus\pi\mathfrak{O}_\nu}} f\begin{pmatrix} \pi x_1 \\ (c_{2,3}x_3+i_k)+\dots \\ \pi^{Z_{2,3}}x_3 \end{pmatrix} f(\xi(-c_{1,3}x_3))\mathrm{d}\mathcal{X} \\
&=: M_3^{(\xi)}(i_k) + M_4^{(\xi)}(i_k) \qquad \text{defined respectively.}
\end{aligned}$$

By Lemma 5.5.1 the first of these integrals $M_3^{(\xi)}(i_k)$ is found to satisfy,

$$\sum_{i_k\in\Pi\setminus 0} M_3^{(\xi)}(i_k) = \frac{(\rho-1)}{m} - \sum_{x_3\in\Pi\setminus 0} f(c_{2,3}x_3)f(-\xi c_{1,3}x_3).$$

The second integral $M_4^{(\xi)}(i_k)$ requires us to first make the substitution,

$$x_3 \longmapsto -c_{2,3}^{-1}i_k + \pi x_3,$$

followed by the change of variable,

$$x_3 \longmapsto x_3 - c_{2,3}^{-1}i_{k+1} - \dots - \pi^{Z_{1,2}-k-1}c_{2,3}^{-1}x_2.$$

Having done this we are then able to conclude that,

$$\begin{aligned}
M_4^{(\xi)}(i_k) &= \int_{\mathfrak{O}_\nu^3} f\begin{pmatrix} \pi x_1 \\ \pi c_{2,3}x_3 \\ -\pi^{Z_{2,3}}c_{2,3}^{-1}i_k \end{pmatrix} f(\xi c_{1,3}c_{2,3}^{-1}i_k)\mathrm{d}\mathcal{X} \\
&= f(-c_{2,3}^{-1}i_k)f(\xi c_{1,3}c_{2,3}^{-1}i_k) \quad \text{by Lemma 6.3.1.}
\end{aligned}$$

Therefore we have found that whenever $0 \le Z_{1,3} - Z_{2,3} \le k$ the integral $L_3^{(\xi)}(i_k)$ satisfies,

$$L_3^{(\xi)}(i_k) = M_3^{(\xi)}(i_k) + M_4^{(\xi)}(i_k) + M_2^{(\xi)}(i_k).$$

Let us now return to the calculation of the integral $L_4^{(\xi)}(i_k)$ given by,

$$L_4^{(\xi)}(i_k) = \int_{\pi^{k+1}\mathfrak{O}_\nu} \int_{\mathfrak{O}_\nu} \int_{\mathfrak{O}_\nu} f\left( \begin{smallmatrix} x_1 + c_{1,3}x_3 \\ .+\pi^k i_k + \pi^{Z_{1,3}-Z_{2,3}} c_{2,3}x_3 \\ \pi^{Z_{1,3}} x_3 \end{smallmatrix} \right) f\left( \begin{smallmatrix} \xi\pi^k i_k \\ \xi\pi^{Z_{1,3}} x_3 \end{smallmatrix} \right) \mathrm{d}\mathcal{X}.$$

Since this integral will further depend on which of $k$ and $Z_{1,3}$ is greatest we must again split this case into two. In each of the cases we calculate,

$k \ge Z_{1,3}$

$$\begin{aligned} L_4^{(\xi)}(i_k) &= \int_{\pi^{k+1}\mathfrak{O}_\nu} \int_{\mathfrak{O}_\nu} \int_{\pi^{k-Z_{1,3}}\mathfrak{O}_\nu \setminus \pi^{k+1}\mathfrak{O}_\nu} f(\pi^{Z_{1,3}}x_3) f(\xi i_k) \mathrm{d}\mathcal{X} + f(i_k) f(\xi i_k) \int_{\pi^{k+1}\mathfrak{O}_\nu} \int_{\mathfrak{O}_\nu} \int_{\pi^{k+1}\mathfrak{O}_\nu} \mathrm{d}\mathcal{X} \\ &= f(\xi i_k) \frac{(\rho^{Z_{1,3}+1} - 1)}{m}. \end{aligned}$$

$k < Z_{1,3}$

$$\begin{aligned} L_4^{(\xi)}(i_k) &= \int_{\pi^{k+1}\mathfrak{O}_\nu} \int_{\mathfrak{O}_\nu} \int_{\mathfrak{O}_\nu \setminus \pi^{k+1}\mathfrak{O}_\nu} f(\pi^{Z_{1,3}}x_3) f(\xi i_k) \mathrm{d}\mathcal{X} + f(i_k) f(\xi i_k) \int_{\pi^{k+1}\mathfrak{O}_\nu} \int_{\mathfrak{O}_\nu} \int_{\pi^{k+1}\mathfrak{O}_\nu} \mathrm{d}\mathcal{X} \\ &= f(\xi i_k) \frac{(\rho^{k+1} - 1)}{m}. \end{aligned}$$

Having calculated all of the possible values of $L_1^{(\xi)}(i_k)$ and $L_2^{(\xi)}(i_k)$ we find that we shall have three possible distinct cases for the integral $I_1^{(\xi)}(i_k)$. They are as follows:

**Case (2.1):** $0 \le k < Z_{1,3} - Z_{2,3} \le Z_{1,3}$

In this case we found that the integral $L_2^{(\xi)}(i_k)$ was trivial. Therefore we simply have,

$$\sum_{i_k \in \Pi \setminus 0} I_1^{(\xi)}(i_k) = \sum_{i_k \in \Pi \setminus 0} L_1^{(\xi)}(i_k) = \frac{(\rho - 1)}{m}(Z_{1,3} - k)$$

from which we may deduce,

$$\prod_{\xi \in \mu_m} \xi^{\sum_{i_k} I_1^{(\xi)}(i_k)} = (-1)^{\frac{(\rho-1)}{2^r}(Z_{1,3}-k)}.$$

**Case (2.2):** $\quad 0 \leq Z_{1,3} - Z_{2,3} \leq k < Z_{1,3}$

For this case neither of the integrals $L_1^{(\xi)}(i_k)$ or $L_2^{(\xi)}(i_k)$ are trivial. Using the results we found in this section we now have,

$$\begin{aligned}
\sum_{i_k \in \Pi \backslash 0} I_1^{(\xi)}(i_k) &= \sum_{i_k \in \Pi \backslash 0} L_1^{(\xi)}(i_k) + M_3^{(\xi)}(i_k) + M_4^{(\xi)}(i_k) + M_2^{(\xi)}(i_k) + L_4^{(\xi)}(i_k) \\
&= \frac{(\rho-1)}{m}(Z_{1,3} - k) + \frac{(\rho-1)}{m} - \sum_{x_3 \in \Pi \backslash 0} f(c_{2,3}x_3) f(-\xi c_{1,3} x_3) \\
&\quad + \sum_{i_k \in \Pi \backslash 0} f(-c_{2,3}^{-1} i_k) f(\xi c_{1,3} c_{2,3}^{-1} i_k) + \frac{(\rho-1)}{m}(Z_{1,3} - Z_{2,3}) + \frac{(\rho-1)}{m}(k+1)
\end{aligned}$$

Therefore, whenever $0 \leq Z_{1,3} - Z_{2,3} \leq k < Z_{1,3}$ the integral $I_1^{(\xi)}(i_k)$ satisfies,

$$\begin{aligned}
\prod_{\xi \in \mu_m} \xi^{\sum_{i_k} I_1^{(\xi)}(i_k)} &= (-1)^{\frac{(\rho-1)}{2^r}(Z_{1,3}-k)} (-1)^{\frac{(\rho-1)}{2^r}} (-c_{1,3}/c_{2,3}, \pi)_{\nu,m} (-c_{1,3}^{-1}, \pi)_{\nu,m} \\
&\quad .(-1)^{\frac{(\rho-1)}{2^r}(Z_{1,3}-Z_{2,3})} (-1)^{\frac{(\rho-1)}{2^r}(k+1)} \\
&= (c_{2,3}^{-1}, \pi)_{\nu,m} \, (-1)^{\frac{(\rho-1)}{2^r} Z_{2,3}}.
\end{aligned}$$

**Case (2.3):** $\quad 0 \leq Z_{1,3} - Z_{2,3} \leq Z_{1,3} \leq k$

Using our previous results for this case we find,

$$\begin{aligned}
\sum_{i_k \in \Pi \backslash 0} I_1^{(\xi)}(i_k) &= \sum_{i_k \in \Pi \backslash 0} M_3^{(\xi)}(i_k) + M_4^{(\xi)}(i_k) + M_2^{(\xi)}(i_k) + L_4^{(\xi)}(i_k) \\
&= \frac{(\rho-1)}{m} - \sum_{x_3 \in \Pi \backslash 0} f(c_{2,3}x_3) f(-\xi c_{1,3} x_3) \\
&\quad + \sum_{i_k \in \Pi \backslash 0} f(-c_{2,3}^{-1} i_k) f(\xi c_{1,3} c_{2,3}^{-1} i_k) + \frac{(\rho-1)}{m}(Z_{1,3} - Z_{2,3}) + \frac{(\rho-1)}{m}(Z_{1,3}+1),
\end{aligned}$$

which allows us to deduce that,

$$\begin{aligned}
\prod_{\xi \in \mu_m} \xi^{\sum_{i_k} I_1^{(\xi)}(i_k)} &= (-1)^{\frac{(\rho-1)}{2^r}} (-c_{1,3}/c_{2,3}, \pi)_{\nu,m} (-c_{1,3}^{-1}, \pi)_{\nu,m} \\
&\quad .(-1)^{\frac{(\rho-1)}{2^r}(Z_{1,3}-Z_{2,3})} (-1)^{\frac{(\rho-1)}{2^r}(Z_{1,3}+1)} \\
&= (c_{2,3}^{-1}, \pi)_{\nu,m} \, (-1)^{\frac{(\rho-1)}{2^r} Z_{2,3}}.
\end{aligned}$$

**Remark:**

Before we move on let us first note that we may further simplify our results for $I_1^{(\xi)}(i_k)$ into just two cases. This is done by noticing that in Cases (1), (2.2) and (2.3) we shall always have $Z_{1,3} - Z_{2,3} \leq k$, and that Case (2.1) is the only one to have $k < Z_{1,3} - Z_{2,3}$.

Therefore, having re-named these Cases (1*) and (2*) respectively, we simply find

•Case (1*) : $Z_{1,3} - Z_{2,3} \leq k$

$$\prod_{\xi \in \mu_m} \xi^{\sum_{i_k} I_1^{(\xi)}(i_k)} = (c_{2,3}^{-1}, \pi)_{\nu,m} \, (-1)^{\frac{(\rho-1)}{2^r} Z_{2,3}}.$$

•Case (2*) : $k < Z_{1,3} - Z_{2,3}$

$$\prod_{\xi \in \mu_m} \xi^{\sum_{i_k} I_1^{(\xi)}(i_k)} = (-1)^{\frac{(\rho-1)}{2^r}(Z_{1,3}-k)}.$$

**The Integrals $I_2^{(\xi)}(i_k)$**

Having made use of the substitutions relating to the dissection with respect to $n_{1,2}$ we are able to write,

$$I_2^{(\xi)}(i_k) = \int_{\mathfrak{O}_\nu^3} f\begin{pmatrix} x_1+\pi^{-Z_{1,2}}c_{1,2}(\pi^{Z_{1,2}}x_2+\ldots+\pi^k i_k)+\pi^{-Z_{1,3}}c_{1,3}x_3 \\ (\pi^{Z_{1,2}}x_2+\ldots+\pi^k i_k)+\pi^{-Z_{2,3}}c_{2,3}x_3 \\ x_3 \end{pmatrix} f(\xi c_{1,2} i_k)\, \mathrm{d}\mathcal{X}.$$

We shall begin this calculation by first considering the two cases $Z_{1,3} < Z_{2,3}$ and $Z_{1,3} \geq Z_{2,3}$. By doing this we may concentrate on the similarities between this result and the previous results for $I_2^{(\xi)}(i_k)$ in the calculations of $dec_\nu(n_{2,3}, n_{1,2})$ and $dec_\nu(n_{1,3}, n_{1,2})$.

**Case (1)** $Z_{1,3} < Z_{2,3}$ :

We shall begin by considering the case when $Z_{1,3} < Z_{2,3}$. Here, unlike with the previous integral $I_1^{(\xi)}(i_k)$, we do in fact find that the function $f$ simply satisfies,

$$f\begin{pmatrix} x_1+\pi^{-Z_{1,2}}c_{1,2}(.+\pi^k i_k)+\pi^{-Z_{1,3}}c_{1,3}x_3 \\ (\pi^{Z_{1,2}}x_2+\ldots+\pi^k i_k)+\pi^{-Z_{2,3}}c_{2,3}x_3 \\ x_3 \end{pmatrix} = f\begin{pmatrix} x_1+\pi^{-Z_{1,2}}c_{1,2}(.+\pi^k i_k) \\ (\pi^{Z_{1,2}}x_2+\ldots+\pi^k i_k)+\pi^{-Z_{2,3}}c_{2,3}x_3 \\ x_3 \end{pmatrix}.$$

Therefore, in the case that $Z_{1,3} < Z_{2,3}$ we find that the integral $I_2^{(\xi)}(i_k)$ is identical to the integral $I_2^{(\xi)}(i_k)$ given in the calculation of $dec_\nu(n_{2,3}, n_{1,2})$ on page 169.

Therefore the integral $I_2^{(\xi)}(i_k)$ satisfies,

**Case (1.1):** $k \leq Z_{1,2} - Z_{2,3}$

$$\prod_{\xi \in \mu_m} \xi^{\sum_{i_k} - I_2^{(\xi)}(i_k)} = 1$$

**Case (1.2):** $k > Z_{1,2} - Z_{2,3}$

$$\prod_{\xi \in \mu_m} \xi^{\sum_{i_k} - I_2^{(\xi)}(i_k)} = (-1)^{\frac{(\rho-1)}{2^r}(Z_{1,2}+Z_{2,3})}(-1)^{\frac{(\rho-1)}{2^r}k}.$$

**Case (2)** $\quad Z_{1,3} \geq Z_{2,3}$ :

Let us now suppose that $Z_{1,3} \geq Z_{2,3}$. Unfortunately in this case we do not always find,

$$f\begin{pmatrix} x_1+\pi^{-Z_{1,2}}c_{1,2}(.+\pi^k i_k)+\pi^{-Z_{1,3}}c_{1,3}x_3 \\ (\pi^{Z_{1,2}}x_2+...+\pi^k i_k)+\pi^{-Z_{2,3}}c_{2,3}x_3 \\ x_3 \end{pmatrix} \approx f\begin{pmatrix} x_1+\pi^{-Z_{1,2}}c_{1,2}(.+\pi^k i_k)+\pi^{-Z_{1,3}}c_{1,3}x_3 \\ (\pi^{Z_{1,2}}x_2+...+\pi^k i_k) \\ x_3 \end{pmatrix}.$$

However, if this statement were true, we would have found that this integral $I_2^{(\xi)}(i_k)$ was identical to that given in the calculation of $dec_\nu(n_{1,3}, n_{1,2})$ on page 163. Therefore we shall once again return to that calculation and see what changes the addition of the term $(\pi^{-Z_{2,3}}c_{2,3}x_3)$ makes.

**Case (2.1)** $\quad k < Z_{1,2} - Z_{1,3}$ :

Whenever $k < Z_{1,2} - Z_{1,3}$ we do indeed find,

$$f\begin{pmatrix} x_1+\pi^{-Z_{1,2}}c_{1,2}(.+\pi^k i_k)+\pi^{-Z_{1,3}}c_{1,3}x_3 \\ (\pi^{Z_{1,2}}x_2+...+\pi^k i_k)+\pi^{-Z_{2,3}}c_{2,3}x_3 \\ x_3 \end{pmatrix} = f\begin{pmatrix} x_1+\pi^{-Z_{1,2}}c_{1,2}(.+\pi^k i_k)+\pi^{-Z_{1,3}}c_{1,3}x_3 \\ (\pi^{Z_{1,2}}x_2+...+\pi^k i_k) \\ x_3 \end{pmatrix}.$$

Therefore, the integral $I_2^{(\xi)}(i_k)$ is indeed identical to that given in the calculation of $dec_\nu(n_{1,3}, n_{1,2})$ on page 163. Referring to our previous result we are able to write,

$$\prod_{\xi \in \mu_m} \xi^{\sum_{i_k} - I_2^{(\xi)}(i_k)} = 1.$$

**Case (2.2)** $\quad k \geq Z_{1,2} - Z_{1,3}$ :

Let us now consider the case when $k \geq Z_{1,2} - Z_{1,3}$. Following through the calculation for $I_2^{(\xi)}(i_k)$ on page 163, we see that the additional term $\pi^{-Z_{2,3}}c_{2,3}x_3$ will have no effect until we reach the integral $L_2^{(\xi)}(i_k)$. That is, we are still able to write

$$\prod_{\xi \in \mu_m} \xi^{\sum_{i_k} - I_2^{(\xi)}(i_k)} = (-1)^{\frac{(\rho-1)}{2^r}\left(k-(Z_{1,2}-Z_{1,3})\right)}(-1)^{\frac{(\rho-1)}{2^r}}.\prod_{\xi \in \mu_m} \xi^{\sum_{i_k} - L_2^{(\xi)}(i_k)}. \tag{6.4}$$

However, returning to the point (♠) on page 164, we see that for this case the integral $L_2^{(\xi)}(i_k)$ is now given by,

$$L_2^{(\xi)}(i_k) = f(\xi c_{1,2} i_k) \int_{\mathfrak{O}_\nu^3} f\left( \begin{array}{c} \pi c_{1,3} x_3 \\ \pi^{Z_{1,2}} i_k - \pi^{(Z_{1,3}-Z_{2,3})} c_{2,3} c_{1,2} c_{1,3}^{-1} i_k + \dots \\ -\pi^{Z_{1,3}} c_{1,2} c_{1,3}^{-1} i_k \end{array} \right) \mathrm{d}\mathcal{X}, \tag{6.5}$$

and will no longer just depend on which of $Z_{1,2}$ and $Z_{1,3}$ is greatest.

Instead this integral will now depend on the values of $Z_{1,2}$ and $(Z_{1,3} - Z_{2,3})$. In the case that $Z_{1,2} = (Z_{1,3} - Z_{2,3})$ the integral will further depend on the constant $(1 - c_{2,3} c_{1,2} c_{1,3}^{-1})$. Therefore, we shall once again split this into the following cases.

**Case (2.2.1)** $\quad Z_{1,2} < (Z_{1,3} - Z_{2,3})$ :

Whenever we have $Z_{1,2} < (Z_{1,3} - Z_{2,3})$ the integral $L_2^{(\xi)}(i_k)$ is identical to that given on page 164 in the case when $Z_{1,2} < Z_{1,3}$. Referring to that result we may conclude that,

$$L_2^{(\xi)}(i_k) = f(\xi c_{1,2} i_k) \frac{(\rho^{Z_{1,2}} - 1)}{m} + f(i_k) f(\xi c_{1,2} i_k).$$

Thus, using equation (6.4), whenever we have $Z_{1,2} < (Z_{1,3} - Z_{2,3})$ the integral $I_2^{(\xi)}(i_k)$ satisfies,

$$\begin{aligned} & Z_{1,2} < (Z_{1,3} - Z_{2,3}) \\ & \qquad \prod_{\xi \in \mu_m} \xi^{\sum_{i_k} - I_2^{(\xi)}(i_k)} = (-1)^{\frac{(\rho-1)}{2^r}(k-(Z_{1,2}-Z_{1,3}))} . (-1)^{\frac{(\rho-1)}{2^r}} . (-1)^{\frac{(\rho-1)}{2^r} Z_{1,2}} . (c_{1,2}, \pi)_{\nu,m} \\ & \qquad \qquad \qquad \qquad = (-1)^{\frac{(\rho-1)}{2^r}(k+1)} . (-1)^{\frac{(\rho-1)}{2^r} Z_{1,3}} . (c_{1,2}, \pi)_{\nu,m}. \end{aligned}$$

**Case (2.2.2)** $\quad Z_{1,2} > (Z_{1,3} - Z_{2,3})$ :

For this case we simply note that the calculation of $L_2^{(\xi)}(i_k)$ is essentially the same as for the previous case. Therefore, by exchanging the constants

$$Z_{1,2} \longmapsto (Z_{1,3} - Z_{2,3}) \qquad \text{and} \qquad i_k \longmapsto -c_{2,3} c_{1,2} c_{1,3}^{-1} i_k,$$

for the terms within the integral and using our previous result, we are immediately able to conclude that

$$L_2^{(\xi)}(i_k) = f(\xi c_{1,2} i_k) \frac{(\rho^{(Z_{1,3}-Z_{2,3})} - 1)}{m} + f(-c_{2,3} c_{1,2} c_{1,3}^{-1} i_k) f(\xi c_{1,2} i_k).$$

Therefore, using equation (6.4), for this case we find

$Z_{1,2} > (Z_{1,3} - Z_{2,3})$

$$\prod_{\xi\in\mu_m} \xi^{\sum i_k - I_2^{(\xi)}(i_k)} = (-1)^{\frac{(\rho-1)}{2^r}(k-(Z_{1,2}-Z_{1,3}))}.(-1)^{\frac{(\rho-1)}{2^r}}.(-1)^{\frac{(\rho-1)}{2^r}(Z_{1,3}-Z_{2,3})}.(-c_{1,3}/c_{2,3},\pi)_{\nu,m}$$
$$= (-1)^{\frac{(\rho-1)}{2^r}\left(k-(Z_{1,2}-Z_{2,3})\right)}.(c_{1,3}/c_{2,3},\pi)_{\nu,m}.$$

**Case (2.2.3)** $\quad Z_{1,2} = (Z_{1,3} - Z_{2,3})$ **:**

For this case we must now go back to equation (6.5) and consider the terms which have been discarded. By considering the full expression, when $Z_{1,2} = (Z_{1,3} - Z_{2,3})$, we find that the integral $L_2^{(\xi)}(i_k)$ satisfies,

$$L_2^{(\xi)}(i_k) = f(\xi c_{1,2} i_k)\int_{\mathfrak{O}_\nu^3} f\left(\begin{matrix} c_{1,3}x_3 \\ \left(\begin{matrix}(\pi^{2Z_{1,2}-k-1}x_2+.+\pi^{Z_{1,2}-1}i_k)-\pi^{(Z_{1,3}-Z_{2,3})-1}c_{2,3}c_{1,2}c_{1,3}^{-1}i_k+\pi^{(Z_{1,3}-Z_{2,3})}c_{2,3}x_3 \\ -\pi^{(Z_{1,3}-Z_{2,3})}(c_{2,3}c_{1,2}c_{1,3}^{-1}i_{k+1}-.-\pi^{Z_{1,2}-k}c_{2,3}c_{1,3}^{-1}(x_1+c_{1,2}x_2))\end{matrix}\right) \\ -\pi^{Z_{1,3}-1}c_{1,2}c_{1,3}^{-1}i_k\end{matrix}\right) d\mathcal{X}$$
$$= f(\xi c_{1,2} i_k)\int_{\mathfrak{O}_\nu^3} f\left(\begin{matrix} c_{1,3}x_3 \\ \pi^{Z_{1,2}+E-1}e(i_k+\pi i_{k+1}+\ldots+\pi^{Z_{1,2}-k+1}(c_{1,2}^{-1}x_1+x_2))+\pi^{Z_{1,2}}c_{2,3}x_3 \\ -\pi^{Z_{1,3}-1}c_{1,2}c_{1,3}^{-1}i_k\end{matrix}\right) d\mathcal{X},$$

where we have defined,

$$(1 - c_{2,3}c_{1,2}c_{1,3}^{-1}) = \pi^E e,$$

for some $E \in \mathbb{Z}^{\geq 0}$ and where either $e = 0$ or $\mid e \mid_\nu = 1$. Now this integral $L_2^{(\xi)}(i_k)$ will clearly depend on the values of $E$ and $e$.

**Suppose $e = 0$:**

In this case we simply find,

$$L_2^{(\xi)}(i_k) = f(\xi c_{1,2} i_k)\int_{\mathfrak{O}_\nu^3} f\left(\begin{matrix} c_{1,3}x_3 \\ \pi^{Z_{1,2}}c_{2,3}x_3 \\ -\pi^{Z_{1,3}-1}c_{1,2}c_{1,3}^{-1}i_k\end{matrix}\right) d\mathcal{X}$$
$$= f(\xi c_{1,2} i_k)\int_{\mathfrak{O}_\nu^3} f\left(\begin{matrix} c_{1,3}x_3 \\ -\pi^{Z_{1,3}-1}c_{1,2}c_{1,3}^{-1}i_k\end{matrix}\right) d\mathcal{X}$$

Splitting the integral up and calculating each of the parts we have,

$$= f(\xi c_{1,2} i_k)\frac{(\rho^{Z_{1,3}}-1)}{m} + f(-c_{1,2}c_{1,3}^{-1}i_k)f(\xi c_{1,2} i_k).$$

Finally we are able to deduce that,

$$\prod_{\xi\in\mu_m} \xi^{\sum i_k - I_2^{(\xi)}(i_k)} = (-1)^{\frac{(\rho-1)}{2^r}(k-(Z_{1,2}-Z_{1,3}))}.(-1)^{\frac{(\rho-1)}{2^r}}.(-1)^{\frac{(\rho-1)}{2^r}Z_{1,3}}.(-c_{1,3}^{-1},\pi)_{\nu,m}$$
$$= (c_{1,3},\pi)_{\nu,m}.(-1)^{\frac{(\rho-1)}{2^r}k}.(-1)^{\frac{(\rho-1)}{2^r}Z_{1,2}}.$$

**Suppose $e \neq 0$:**

Here we find that the integral $L_2^{(\xi)}(i_k)$ satisfies,

$$L_2^{(\xi)}(i_k) = f(\xi c_{1,2} i_k) \int_{\mathfrak{O}_\nu} \int_{\mathfrak{O}_\nu} \int_{\mathfrak{O}_\nu} f \begin{pmatrix} c_{1,3} x_3 \\ \pi^{Z_{1,2}+E-1} e(i_k + \pi i_{k+1} + \ldots + \pi^{Z_{1,2}-k+1}(x_1)) + \pi^{Z_{1,2}} c_{2,3} x_3 \\ -\pi^{Z_{1,3}-1} c_{1,2} c_{1,3}^{-1} i_k \end{pmatrix} \mathrm{d}\mathcal{X}$$
$$= f(\xi c_{1,2} i_k) \int_{\mathfrak{O}_\nu} \int_{\mathfrak{O}_\nu} \int_{\mathfrak{O}_\nu} f \begin{pmatrix} c_{1,3} x_3 \\ \pi^{Z_{1,2}+E-1} e i_k \\ -\pi^{Z_{1,3}-1} c_{1,2} c_{1,3}^{-1} i_k \end{pmatrix} \mathrm{d}\mathcal{X}.$$

Clearly this integral will split into two cases depending on which of,

$$Z_{1,2} + E = Z_{1,3} - (Z_{2,3} - E) \quad \text{and} \quad Z_{1,3}$$

is greatest. In each case, by splitting the integral and calculating the parts, we immediately find,

- $E \leq Z_{2,3}$ : $\quad L_2^{(\xi)}(i_k) = f(\xi c_{1,2} i_k) \dfrac{(\rho^{(Z_{1,2}+E)} - 1)}{m} + f(e i_k) f(\xi c_{1,2} i_k).$

$$\Rightarrow \quad \prod_{\xi \in \mu_m} \xi^{\sum_{i_k} - I_2^{(\xi)}(i_k)} = (-1)^{\frac{(\rho-1)}{2^r}(k-(Z_{1,2}-Z_{1,3}))}.(-1)^{\frac{(\rho-1)}{2^r}}.(-1)^{\frac{(\rho-1)}{2^r}(Z_{1,2}+E)}.(c_{1,2}/e, \pi)_{\nu,m}$$
$$= (c_{1,2}/e, \pi)_{\nu,m}.(-1)^{\frac{(\rho-1)}{2^r}(k+1)}.(-1)^{\frac{(\rho-1)}{2^r}(Z_{1,3}+E)}.$$

- $E > Z_{2,3}$ :

$$L_2^{(\xi)}(i_k) = f(\xi c_{1,2} i_k) \frac{(\rho^{Z_{1,3}} - 1)}{m} + f(-c_{1,2} c_{1,3}^{-1} i_k) f(\xi c_{1,2} i_k).$$

$$\Rightarrow \quad \prod_{\xi \in \mu_m} \xi^{\sum_{i_k} - I_2^{(\xi)}(i_k)} = (-1)^{\frac{(\rho-1)}{2^r}(k-(Z_{1,2}-Z_{1,3}))}.(-1)^{\frac{(\rho-1)}{2^r}}.(-1)^{\frac{(\rho-1)}{2^r} Z_{1,3}}.(-c_{1,3}, \pi)_{\nu,m}$$
$$= (c_{1,3}, \pi)_{\nu,m}.(-1)^{\frac{(\rho-1)}{2^r} k}.(-1)^{\frac{(\rho-1)}{2^r} Z_{1,2}}.$$

By looking at the results given above we see that when $Z_{1,2} = (Z_{1,3} - Z_{2,3})$ we in fact have only two cases. These are,

**Case (2.2.3.1):** $E \leq Z_{2,3}$, **Case (2.2.3.2):** $E > Z_{2,3}$ or $e = 0$

**Remark:**

Before we continue let us note that we have now seen seven distinct results for the integrals $I_2^{(\xi)}(i_k)$. To clarify, the various cases are as follows:

$$\begin{cases} (1){:}Z_{1,3} < Z_{2,3} & \begin{cases} (1.1){:}k \leq Z_{1,2} - Z_{2,3} \\ (1.2){:}Z_{1,2} - Z_{2,3} < k \end{cases} \\ \\ (2){:}Z_{1,3} \geq Z_{2,3} & \begin{cases} (2.1){:}k < Z_{1,2} - Z_{1,3} \\ \\ (2.2){:}Z_{1,2} - Z_{1,3} \leq k & \begin{cases} (2.2.1){:}Z_{1,2} < (Z_{1,3} - Z_{2,3}) \\ (2.2.2){:}(Z_{1,3} - Z_{2,3}) < Z_{1,2} \\ \\ (2.2.3){:}Z_{1,2} = (Z_{1,3} - Z_{2,3}) & \begin{cases} (2.2.3.1){:}E \leq Z_{2,3} \\ (2.2.3.2){:}E > Z_{2,3}. \end{cases} \end{cases} \end{cases} \end{cases}$$

**Results for the cocycle $dec_\nu(n_{2,3}n_{1,3}, n_{1,2})$**

We are now able to go back to our results for the integrals $I_1^{(\xi)}(i_k)$ and $I_2^{(\xi)}(i_k)$ and put them together to prove the statement of our theorem. After considering the various cases which present themselves we see that we are able to conclude this proof by looking at seven distinct cases. They are as follows:

**Results for** $Z_{1,3} - Z_{2,3} < 0 < Z_{2,3} \leq Z_{1,2}$ **:**

$$\begin{aligned} dec_\nu(n_{2,3}n_{1,3}, n_{1,2}) &= \prod_{k=0}^{Z_{1,3}-1} \prod_{\xi \in \mu_m} \xi^{\sum_{i_k} I_1^{(\xi)}(i_k)} . \prod_{k=0}^{Z_{1,3}-1} \prod_{\xi \in \mu_m} \xi^{-\sum_{i_k} I_2^{(\xi)}(i_k)} \\ &= \prod_{k=0}^{Z_{1,2}-1} (1^*) . \prod_{k=0}^{(Z_{1,2}-Z_{2,3})} (1.1) \prod_{k=(Z_{1,2}-Z_{2,3}+1)}^{Z_{1,2}-1} (1.2) \\ &= (-1)^{\frac{(\rho-1)}{2^r} Z_{2,3} Z_{1,2}} (c_{2,3}^{-1}, \pi)_{\nu,m}^{Z_{1,2}} . (-1)^{\frac{(\rho-1)}{2^r} \frac{Z_{2,3}(Z_{2,3}-1)}{2}} \\ &= (c_{2,3}, \pi)_{\nu,m}^{-Z_{1,2}} (-1)^{\frac{(\rho-1)}{2^r} Z_{2,3} Z_{1,2}} (-1)^{\frac{(\rho-1)}{2^r} \frac{Z_{2,3}(Z_{2,3}-1)}{2}} . \end{aligned}$$

**Results for** $Z_{1,3} - Z_{2,3} < 0 < Z_{1,2} < Z_{2,3}$ :

$$dec_\nu(n_{2,3}n_{1,3}, n_{1,2}) = \prod_{k=0}^{Z_{1,2}-1} (1^*). \prod_{k=0}^{Z_{1,2}-1} (1.2)$$

$$= (-1)^{\frac{(\rho-1)}{2^r} Z_{2,3}Z_{1,2}} (c_{2,3}^{-1}, \pi)_{\nu,m}^{Z_{1,2}} .(-1)^{\frac{(\rho-1)}{2^r} Z_{1,2}Z_{2,3}} (-1)^{\frac{(\rho-1)}{2^r} \frac{Z_{1,2}(Z_{1,2}+1)}{2}}$$

$$= (c_{2,3}, \pi)_{\nu,m}^{-Z_{1,2}} (-1)^{\frac{(\rho-1)}{2^r} \frac{Z_{1,2}(Z_{1,2}+1)}{2}}.$$

**Results for** $0 \le Z_{1,3} - Z_{2,3} \le Z_{1,3} < Z_{1,2}$ :

$$dec_\nu(n_{2,3}n_{1,3}, n_{1,2}) = \prod_{k=0}^{(Z_{1,3}-Z_{2,3}-1)} (2^*) \prod_{k=(Z_{1,3}-Z_{2,3})}^{Z_{1,2}-1} (1^*). \prod_{k=0}^{(Z_{1,2}-Z_{1,3})-1} (2.1) \prod_{k=(Z_{1,2}-Z_{1,3})}^{Z_{1,2}-1} (2.2.2)$$

$$= (-1)^{\frac{(\rho-1)}{2^r} Z_{2,3}(Z_{1,2}+Z_{1,3})} (-1)^{\frac{(\rho-1)}{2^r} \frac{Z_{1,3}(Z_{1,3}+1)}{2}} (-1)^{\frac{(\rho-1)}{2^r} \frac{Z_{2,3}(Z_{2,3}-1)}{2}} (c_{2,3}^{-1}, \pi)_{\nu,m}^{Z_{1,2}-(Z_{1,3}-Z_{2,3})}$$

$$.(-1)^{\frac{(\rho-1)}{2^r} Z_{1,3}Z_{2,3}} (-1)^{\frac{(\rho-1)}{2^r} \frac{Z_{1,3}(Z_{1,3}+1)}{2}} (c_{2,3}^{-1}c_{1,3}, \pi)_{\nu,m}^{Z_{1,3}}$$

$$= (c_{2,3}, \pi)_{\nu,m}^{-(Z_{1,2}+Z_{2,3})} (c_{1,3}, \pi)_{\nu,m}^{Z_{1,3}} (-1)^{\frac{(\rho-1)}{2^r} Z_{2,3}Z_{1,2}} (-1)^{\frac{(\rho-1)}{2^r} \frac{Z_{2,3}(Z_{2,3}-1)}{2}}.$$

**Results for** $0 \le Z_{1,3} - Z_{2,3} < Z_{1,2} \le Z_{1,3}$ :

$$dec_\nu(n_{2,3}n_{1,3}, n_{1,2}) = \prod_{k=0}^{(Z_{1,3}-Z_{2,3}-1)} (2^*) \prod_{k=(Z_{1,3}-Z_{2,3})}^{Z_{1,2}-1} (1^*). \prod_{k=0}^{Z_{1,2}-1} (2.2.2)$$

$$= (-1)^{\frac{(\rho-1)}{2^r} Z_{2,3}(Z_{1,2}+Z_{1,3})} (-1)^{\frac{(\rho-1)}{2^r} \frac{Z_{1,3}(Z_{1,3}+1)}{2}} (-1)^{\frac{(\rho-1)}{2^r} \frac{Z_{2,3}(Z_{2,3}-1)}{2}} (c_{2,3}^{-1}, \pi)_{\nu,m}^{Z_{1,2}-(Z_{1,3}-Z_{2,3})}$$

$$.(-1)^{\frac{(\rho-1)}{2^r} Z_{1,2}Z_{2,3}} (-1)^{\frac{(\rho-1)}{2^r} \frac{Z_{1,2}(Z_{1,2}+1)}{2}} (c_{2,3}^{-1}c_{1,3}, \pi)_{\nu,m}^{Z_{1,2}}$$

$$= (c_{2,3}, \pi)_{\nu,m}^{(Z_{1,3}-Z_{2,3})-2Z_{1,2}} (c_{1,3}, \pi)_{\nu,m}^{Z_{1,2}}$$

$$(-1)^{\frac{(\rho-1)}{2^r} Z_{2,3}Z_{1,3}} (-1)^{\frac{(\rho-1)}{2^r} \frac{Z_{1,2}(Z_{1,2}+1)}{2}} (-1)^{\frac{(\rho-1)}{2^r} \frac{Z_{1,3}(Z_{1,3}+1)}{2}} (-1)^{\frac{(\rho-1)}{2^r} \frac{Z_{2,3}(Z_{2,3}-1)}{2}}.$$

**Results for** $0 < Z_{1,2} < Z_{1,3} - Z_{2,3} \leq Z_{1,3}$ :

$$dec_\nu(n_{2,3}n_{1,3}, n_{1,2}) = \prod_{k=0}^{Z_{1,2}-1} (2^\star). \prod_{k=0}^{Z_{1,2}-1} (2.2.1)$$

$$= (-1)^{\frac{(\rho-1)}{2^r} Z_{1,3}Z_{1,2}}(-1)^{\frac{(\rho-1)}{2^r}\frac{Z_{1,2}(Z_{1,2}-1)}{2}}$$

$$.(-1)^{\frac{(\rho-1)}{2^r} Z_{1,2}Z_{1,3}}(-1)^{\frac{(\rho-1)}{2^r}\frac{Z_{1,2}(Z_{1,2}+1)}{2}}(c_{1,2}, \pi)_{\nu,m}^{Z_{1,2}}$$

$$= (c_{1,2}, \pi)_{\nu,m}^{Z_{1,2}} (-1)^{\frac{(\rho-1)}{2^r} Z_{1,2}}.$$

**Results for** $0 < Z_{1,2} = Z_{1,3} - Z_{2,3} \leq Z_{1,3}$ :

$\circ E \leq Z_{2,3}$ :

$$dec_\nu(n_{2,3}n_{1,3}, n_{1,2}) = \prod_{k=0}^{Z_{1,2}-1} (2^\star). \prod_{k=0}^{Z_{1,2}-1} (2.2.3.1)$$

$$= (-1)^{\frac{(\rho-1)}{2^r} Z_{1,3}Z_{1,2}}(-1)^{\frac{(\rho-1)}{2^r}\frac{Z_{1,2}(Z_{1,2}-1)}{2}}$$

$$.(-1)^{\frac{(\rho-1)}{2^r} Z_{1,2}(Z_{1,3}+E)}(-1)^{\frac{(\rho-1)}{2^r}\frac{Z_{1,2}(Z_{1,2}+1)}{2}}(c_{1,2}/e, \pi)_{\nu,m}^{Z_{1,2}}$$

$$= (c_{1,2}, \pi)_{\nu,m}^{Z_{1,2}} (e, \pi)_{\nu,m}^{-Z_{1,2}} (-1)^{\frac{(\rho-1)}{2^r} Z_{1,2}}(-1)^{\frac{(\rho-1)}{2^r} Z_{1,2}E}$$

$\circ E > Z_{2,3}$ or $e = 0$ :

$$dec_\nu(n_{2,3}n_{1,3}, n_{1,2}) = \prod_{k=0}^{Z_{1,2}-1} (2^\star). \prod_{k=0}^{Z_{1,2}-1} (2.2.3.2)$$

$$= (-1)^{\frac{(\rho-1)}{2^r} Z_{1,3}Z_{1,2}}(-1)^{\frac{(\rho-1)}{2^r}\frac{Z_{1,2}(Z_{1,2}-1)}{2}}.(-1)^{\frac{(\rho-1)}{2^r}\frac{Z_{1,2}(Z_{1,2}+1)}{2}}(c_{1,3}, \pi)_{\nu,m}^{Z_{1,2}}$$

$$= (c_{1,3}, \pi)_{\nu,m}^{Z_{1,2}} (-1)^{\frac{(\rho-1)}{2^r} Z_{1,3}}(-1)^{\frac{(\rho-1)}{2^r} Z_{1,3}Z_{1,2}}.$$

Comparing these results to those stated in the theorem we see that our proof is finally complete.

□

Before we conclude this chapter we should point out that in our latter work we shall in fact be proving that the cocycle $dec_\nu$ splits on the subgroup $N \subset \mathrm{GL}_n(k_\nu)$. This allows us to deduce the following:

**Theorem 6.4.5** *If there exists a cochain $\psi_N$ which splits the cocycle $dec_\nu$ on the subgroup $N \subset \mathrm{GL}_n(k_\nu)$ then the cocycle $dec_\nu$ must satisfy,*

$$dec_\nu(n_1, n_2) = \frac{\psi(n_1)\psi(n_2)}{\psi(n_1\ n_2)} = \frac{\psi(n_2)\psi(n_1)}{\psi(n_2\ n_1)} = dec_\nu(n_2, n_1),$$

*for any matrices $n_1,\ n_2 \in N$ which commute.*

*Furthermore, since we have*

$$n_{2,3}.n_{1,3} = \begin{pmatrix} 1 & 0 & \pi^{-z_{1,3}}c_{1,3} \\ 0 & 1 & \pi^{-z_{2,3}}c_{2,3} \\ 0 & 0 & 1 \end{pmatrix} = n_{1,3}.n_{2,3}$$

$$n_{1,2}.n_{1,3} = \begin{pmatrix} 1 & \pi^{-z_{1,2}}c_{1,2} & \pi^{-z_{1,3}}c_{1,3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = n_{1,3}.n_{1,2},$$

*the cocycle $dec_\nu$ will satisfy,*

$$dec_\nu(n_{1,3}, n_{2,3}) = dec_\nu(n_{2,3}, n_{1,3})$$

$$dec_\nu(n_{1,3}, n_{1,2}) = dec_\nu(n_{1,2}, n_{1,3}).$$

# Chapter 7

# The cocycle $\sigma_n$ on $\mathrm{GL}_n(k_\nu)$

## 7.1 Introduction

As with the previous chapters we begin by letting $k_\nu$ be a local field with valuation $\nu$ and a fixed uniformizing element $\pi$ in $\mathfrak{O}_\nu$ the ring of integers.

In this chapter we shall return to the construction of the metaplectic 2-cocycle $\sigma_n \in Z^2(\mathrm{GL}_n(k_\nu), \mu_m)$, as described in section 1.3 on page 15. Using results given in the work of Banks, Levy and Sepanski [1] we shall be able to explicitly describe the cocycle $\sigma_n$ in much the same way as we have for $dec_\nu$. Throughout this chapter all notation will be as previously described unless stated.

## 7.2 Defining the cocycle $\sigma_n$ on $\mathrm{GL}_n$

Let us recall that in Section 1.3 we had seen an explicit 2-cocycle $\sigma_{SL} \in Z^2(\mathrm{SL}_{n+1}(k_\nu), \mu_m)$ representing the cohomology class in $H^2(\mathrm{SL}_{n+1}(k_\nu), \mu_m)$ of the extension $\tilde{\mathrm{SL}}_{n+1}(k_\nu)$. This cocycle was defined by,

$$\sigma_{SL}(g_1, g_2) = \frac{\mathfrak{s}_{SL_{n+1}}(g_1)\mathfrak{s}_{SL_{n+1}}(g_2)}{\mathfrak{s}_{SL_{n+1}}(g_1 g_2)} \qquad \text{for all } g_1, g_2 \in \mathrm{SL}_{n+1}(k_\nu).$$

Then, by considering the embedding $\imath : \mathrm{GL}_n(k_\nu) \hookrightarrow \mathrm{SL}_{n+1}(k_\nu)$ given by,

$$g \longmapsto \begin{pmatrix} g & 0 \\ 0 & \det(g)^{-1} \end{pmatrix},$$

we were eventually able to define the cocycle $\sigma_n \in Z^2(\mathrm{GL}_n(k_\nu), \mu_m)$ by,

$$\sigma_n(g_1, g_2) = \sigma_{SL_{n+1}}(\imath(g_1), \imath(g_2)).(\det(g_1), \det(g_2))_{\nu,m}^{-1}.$$

As we have seen, this cocycle $\sigma_n \in Z^2(\mathrm{GL}_n(k_\nu), \mu_m)$ corresponds to,

$$1 \to \mu_m \longrightarrow \tilde{\mathcal{G}}_{\sigma_n} \longrightarrow \mathrm{GL}_n(k_\nu) \to 1,$$

the metaplectic cover of $\mathrm{GL}_n(k_\nu)$ by $\mu_m$.

Furthermore, having defined the cocycle, Banks, Levy and Sepanski (Theorem 7, [1]) were able to prove the following result:

**Theorem 1** *The 2-cocycle $\sigma_n$ is the unique element of $Z^2(\mathrm{GL}_n(k_\nu), \mu_m)$ satisfying,*

$$\sigma_n(\alpha_n, \beta_n) = \prod_{i<j} (\tau_i(\alpha_n), \tau_j(\beta_n))_{\nu,m}^{-1} \qquad \textit{for all } \alpha_n,\ \beta_n \in T \subset \mathrm{GL}_n(k_\nu), \tag{7.1}$$

$$\sigma_n(\alpha_n, \eta) = 1 \qquad \textit{for all } \alpha_n \in T,\ \eta \in \mathfrak{M}, \tag{7.2}$$

$$\sigma_n(\eta_1, \eta_2) = 1 \qquad \textit{for all } \eta_1, \eta_2 \in \mathfrak{M} \textit{ such that } l(\eta_1\eta_2) = l(\eta_1) + l(\eta_2), \tag{7.3}$$

$$\sigma_n(\eta, \alpha_n) = \prod_{\substack{\varsigma \in \Phi^+ \\ \eta(\varsigma) < 0}} (-\tau_j(\alpha_n), \tau_i(\alpha_n))_{\nu,m}^{-1} \qquad \textit{for all } \alpha_n \in T,\ \eta \in \mathfrak{M}, \tag{7.4}$$

$$\sigma_n(n, g) = \sigma_n(g, n) = 1 \qquad \textit{for all } n \in N,\ g \in \mathrm{GL}_n(k_\nu), \tag{7.5}$$

$$\sigma_n(n_1 g_1, g_2 n_2) = \sigma_n(g_1, g_2) \qquad \textit{for all } n_i \in N,\ g_i \in \mathrm{GL}_n(k_\nu), \tag{7.6}$$

$$\sigma_n(g_1 n, g_2) = \sigma_n(g_1, n g_2) \qquad \textit{for all } n \in N,\ g_1, g_2 \in \mathrm{GL}_n(k_\nu), \tag{7.7}$$

$$\sigma_n(w_\varsigma, n_\varsigma(x) w_\varsigma) = (x, x)_{\nu,m} \qquad \textit{for all } \varsigma \in \Delta,\ x \in {k_\nu}^\times, \tag{7.8}$$

$$\sigma_n(w_\varsigma, w_\varsigma) = (-1, -1)_{\nu,m} = 1 \qquad \textit{in all cases under consideration.} \tag{7.9}$$

For the remainder of this chapter we shall concentrate on reformulating these statements into the notation we have used when calculating the cocycle $dec_\nu$. We shall also, where appropriate, extend these general statements to more specific results for $\mathrm{GL}_n(k_\nu)$.

## 7.3 Explicit formulae for the cocycle $\sigma_n$

We begin this section by recalling that we have defined,

$$(-1) := \prod_{\xi \in \mu_m} \xi.$$

Once again, this will allow us to calculate the cases when $m$ is odd and $m$ is even simultaneously.

We shall now turn our attention towards finding explicit formulae to describe the cocycle $\sigma_n$ on $\mathrm{GL}_n(k_\nu)$. We shall begin by simply reformulating the statements given in the previous theorem.

**Lemma 5** *For any diagonal matrices $\alpha_n, \beta_n \in T \subset GL_n(k_\nu)$ the cocycle $\sigma_n$ satisfies,*

$$\begin{aligned} \sigma_n(\alpha_n, \beta_n) &= \prod_{i<j} (\tau_i(\alpha_n), \tau_j(\beta_n))_{\nu,m}^{-1} \\ &= \prod_{i<j} (\pi^{X_i} a_i, \pi^{Y_j} b_j)_{\nu,m}^{-1} \\ &= \prod_{i<j} (-1)^{\frac{(\rho-1)}{2r} X_i Y_j} (a_i^{-Y_j} b_j^{X_i}, \pi)_{\nu,m}, \end{aligned}$$

*where $(-1)$ is trivial whenever $m$ is odd.*

**Lemma 6** *For each diagonal matrix $\alpha_n \in T \subset GL_n(k_\nu)$ and each $\eta_{w_1}, \eta_{w_2} \in \mathfrak{M}$ the cocycle $\sigma_n$ satisfies,*

$$\begin{aligned} &(i) && \sigma_n(\alpha_n, \eta_{w_1}) = 1 \\ &(ii) && \sigma_n(\eta_{w_1}, \eta_{w_2}) = 1 \quad \textit{whenever} \quad l(w_1 w_2) = l(w_1) + l(w_2) \\ &(iii) && \sigma_n(w_\varsigma, w_\varsigma) = 1 \quad \textit{whenever} \quad w_\varsigma = \eta_{s_\varsigma} \in \mathfrak{M} \end{aligned}$$

*and finally,*

$$\begin{aligned} (iv) \qquad \sigma_n(\eta_{w_1}, \alpha_n) &= \prod_{\substack{\varsigma \in \Phi^+ \\ \eta_{w_1}(\varsigma) < 0}} (-\tau_j(\alpha_n), \tau_i(\alpha_n))_{\nu,m}^{-1} \\ &= \prod_{\varsigma \in \Phi(w_1)} (-\pi^{X_j} a_j, \pi^{X_i} a_i)_{\nu,m}^{-1} \\ &= \prod_{\varsigma \in \Phi(w_1)} (-1)^{\frac{(\rho-1)}{2r} X_i} (-1)^{\frac{(\rho-1)}{2r} X_i X_j} (a_i^{X_j} a_j^{-X_i}, \pi)_{\nu,m}. \end{aligned}$$

*where $(-1)$ is trivial whenever $m$ is odd.*

Finally, by simply restating (7.5), (7.6) and (7.7) from Theorem 1, for $N \subset \mathrm{GL}_n(k_\nu)$ we also have:

**Lemma 7** *For each $n_1, n_2 \in N$, $g_1, g_2 \in \mathrm{GL}_n(k_\nu)$ the cocycle $\sigma_n$ satisfies,*

$$\sigma_n(n_1, g_1) = \sigma_n(g_1, n_1) = 1$$
$$\sigma_n(n_1 g_1, g_2 n_2) = \sigma_n(g_1, g_2)$$
$$\sigma_n(g_1 n_1, g_2) = \sigma_n(g_1, n_1 g_2).$$

Let us now return to the results given in Theorem 1 and see what else we may deduce. Before we do this however, let us first recall the following notation:

Since the group $W$ is generated by the simple reflections $\{s_\varsigma : \varsigma \in \Delta\}$ each $w \in W$ may be expressed as a minimal product,

$$w = s_{\varsigma_1} \dots s_{\varsigma_\ell}, \qquad \text{with } \varsigma_k = (i_k, i_k + 1) \in \Delta,$$

called a *reduced expression* for $w$ where $l(w) = \ell$ is defined to be the length of $w$. The set $\mathfrak{M}$ is defined to be,

$$\mathfrak{M} = \{\eta_w = w_{\varsigma_1} \dots w_{\varsigma_\ell} : w \in W\}$$

where, $w = s_{\varsigma_1} \dots s_{\varsigma_\ell}$ is some reduced expression for $w \in W$. Let us recall that the map $w \mapsto \eta_w$ is independent of this representation of $w$ as a product of $l(w)$ simple reflections.

In Section 4.6 we saw that $\eta_w = \epsilon_w w$ where, if $w = s_{\varsigma_1} \dots s_{\varsigma_\ell}$ is some reduced expression for $w$, the diagonal matrix $\epsilon_w \in T_{\mathbf{Z}}$ is given by,

$$\epsilon_w = \epsilon_{\varsigma_1} \epsilon_{\varsigma_1}^{s_{\varsigma_1}} \dots \epsilon_{\varsigma_\ell}^{s_{\varsigma_1} \dots s_{\varsigma_{\ell-1}}}.$$

**Corollary 6** *For each $\alpha_n$, $\beta_n \in T \subset GL_n(k_\nu)$ and each $\eta_w \in \mathfrak{M}$ the cocycle $\sigma_n$ satisfies,*

$$\sigma_n(\alpha_n, \beta_n \eta_w) = \sigma_n(\alpha_n, \beta_n).$$

**PROOF:**

By simply applying the cocycle rule we have,

$$\begin{aligned} \sigma_n(\alpha_n, \beta_n \eta_w) &= \sigma_n(\alpha_n, \beta_n) \left( \frac{\sigma_n(\alpha_n \beta_n, \eta_w)}{\sigma_n(\beta_n, \eta_w)} \right) \\ &= \sigma_n(\alpha_n, \beta_n) \quad \text{by equation } (i) \text{ of Lemma 6.} \end{aligned}$$ □

**Corollary 7** *For any $\eta_w$, $w_\varepsilon \in \mathfrak{M}$ the cocycle $\sigma_n$ satisfies,*

$$\sigma_n(\eta_w, w_\varepsilon) = 1.$$

**PROOF:**

Firstly let us recall that $w_\varepsilon = \eta_{s_\varepsilon}$. If $l(ws_\varepsilon) = l(w) + l(s_\varepsilon)$ our result simply follows from equation (*ii*) of Lemma 6. Therefore for the remainder of this proof we shall assume that,

$$\begin{aligned} & l(ws_\varepsilon) \neq l(w) + l(s_\varepsilon) \\ \Rightarrow \quad & l(ws_\varepsilon) = l(w) - 1 \qquad \text{by Lemmas 3 and 4.} \end{aligned} \tag{7.10}$$

Now let $\eta_w = w_{\varsigma_1} \dots w_{\varsigma_\ell} \in \mathfrak{M}$ where $w = s_{\varsigma_1} \dots s_{\varsigma_\ell}$ is some reduced expression for $w \in W$ with $l(w) = \ell$. Then, in order to prove this statement, we shall proceed by induction on the length $l(w)$.

We begin by considering the case when $l(w) = 1$. Then we have $\eta_w = \eta_{s_{\varsigma_1}} = w_{\varsigma_1}$. Now, by (7.10) we must have,

$$l(s_{\varsigma_1} s_\varepsilon) = l(s_{\varsigma_1}) - 1 = 0 = l(1).$$

Therefore, $\varsigma_1 = \varepsilon$ and by equation (*iii*) of Lemma 6 we do indeed find,

$$\sigma_n(\eta_w, w_\varepsilon) = \sigma_n(w_\varepsilon, w_\varepsilon) = 1.$$

Having shown that our result is true when $l(w) = 1$ let us now assume that it is also true for any $\eta_w$ such that $l(w) \leq \ell - 1$.

Now, let us suppose that $\eta_w$ is defined as above with $l(w) = \ell$. Now, once again by (7.10) we must have,

$$l(ws_{\varepsilon_1}) = l(w) - 1 = \ell - 1.$$

Therefore we are free to define $\bar{w} := ws_\varepsilon$ where $\bar{w} = s_{\lambda_1} \dots s_{\lambda_{\ell-1}}$ is some reduced expression for $\bar{w}$. Furthermore, we now have

$$\begin{aligned} \bar{w}s_\varepsilon = w \quad \text{and} \quad l(\bar{w}s_\varepsilon) = l(\bar{w}) + l(s_{\varepsilon_1}) \\ \Rightarrow \eta_w = \eta_{\bar{w}s_\varepsilon} = \eta_{\bar{w}}\eta_{s_\varepsilon} = \eta_{\bar{w}}w_\varepsilon. \end{aligned}$$

So, using the cocycle rule we have,

$$\sigma_n(\eta_w, w_\varepsilon) = \sigma_n(\eta_{\bar{w}}.w_\varepsilon, w_\varepsilon) = \frac{\sigma_n(\eta_{\bar{w}}, w_\varepsilon w_\varepsilon)\sigma_n(w_\varepsilon, w_\varepsilon)}{\sigma_n(\eta_{\bar{w}}, w_\varepsilon)} = \frac{\sigma_n(\eta_{\bar{w}}, \mathcal{I})\sigma_n(w_\varepsilon, w_\varepsilon)}{\sigma_n(\eta_{\bar{w}}, w_\varepsilon)},$$

where $\mathcal{I} \in T_{\mathbb{Z}}$ is some diagonal matrix with entries $\pm 1$.

However, for each term we now find,

$$\begin{aligned} \sigma_n(\eta_{\bar{w}}, \mathcal{I}) = 1 &\quad \text{by equation } (iv), \text{ Lemma 6.} \\ s_n(w_\varepsilon, w_\varepsilon) = 1 &\quad \text{by equation } (iii), \text{ Lemma 6.} \\ \sigma_n(\eta_{\bar{w}}, w_\varepsilon) = 1 &\quad \text{by our inductive hypothesis as } l(\bar{w}) = \ell_1 - 1. \end{aligned}$$

Therefore, in conclusion, we do indeed find that for any $\eta_w \in \mathfrak{M}$ the cocycle $\sigma_n$ satisfies,

$$\sigma_n(\eta_w, w_\varepsilon) = 1.$$

□

**Corollary 8** *For any $\eta_{w_1}, \eta_{w_2} \in \mathfrak{M}$ the cocycle $\sigma_n$ satisfies,*

$$\sigma_n(\eta_{w_1}, \eta_{w_2}) = 1.$$

**PROOF:**

Let us define,

$$\eta_{w_1} = w_{\varsigma_1} \dots w_{\varsigma_{\ell_1}}, \qquad \eta_{w_2} = w_{\varepsilon_1} \dots w_{\varepsilon_{\ell_2}} \quad \in \mathfrak{M},$$

where $w_1 = s_{\varsigma_1} \dots s_{\varsigma_{\ell_1}}$ and $w_2 = s_{\varepsilon_1} \dots s_{\varepsilon_{\ell_2}}$ are reduced expressions for $w_1,\ w_2 \in W$.

Having seen, in the previous Corollary 7, that our result is true whenever we have $l(w_2) = 1$ let us now assume that,

$$\sigma_n(\eta_{w_1}, \eta_w) = 1,$$

for any $\eta_w$ such that $l(w) \leq \ell_2 - 1$ and proceed by induction.

Let $\eta_{w_2}$ be defined as above with $l(w_2) = \ell_2$. Then, using the cocycle rule we have,

$$\begin{aligned} \sigma_n(\eta_{w_1}, \eta_{w_2}) &= \sigma_n(\eta_{w_1}, w_{\varepsilon_1}\ w_{\varepsilon_2} \dots w_{\varepsilon_{\ell_2}}) \\ &= \sigma_n(\eta_{w_1}, w_{\varepsilon_1} \eta_{w_3}) \qquad \text{where} \quad w_3 := s_{\varepsilon_2} \dots s_{\varepsilon_{\ell_2}} \\ &= \frac{\sigma_n(\eta_{w_1} w_{\varepsilon_1}, \eta_{w_3}) \sigma_n(\eta_{w_1}, w_{\varepsilon_1})}{\sigma_n(w_{\varepsilon_1}, \eta_{w_3})}. \end{aligned} \tag{7.11}$$

We shall now deal with each of these terms in turn. To begin we find,

$$\begin{aligned} \sigma_n(\eta_{w_1}, w_{\varepsilon_1}) = 1 &\quad \text{by Corollary 7,} \\ \sigma_n(w_{\varepsilon_1}, \eta_{w_3}) = 1 &\quad \text{by equation } (ii), \text{ Lemma 6.} \end{aligned}$$

Therefore, only one term remains.

Firstly let us suppose that, $l(w_1 s_{\varepsilon_1}) = l(w_1) + l(s_{\varepsilon_1})$. Then,

$$\eta_{w_1} w_{\varepsilon_1} = \eta_{w_1} \eta_{s_{\varepsilon_1}} = \eta_{w_1 s_{\varepsilon_1}},$$

and, using our inductive hypothesis, we are able to write

$$\sigma_n(\eta_{w_1} w_{\varepsilon_1}, \eta_{w_3}) = \sigma_n(\eta_{w_1 s_{\varepsilon_1}}, \eta_{w_3}) = 1,$$

since $l(w_3) = \ell_2 - 1$.

Now let us suppose that $l(w_1 s_{\varepsilon_1}) \neq l(w_1) + l(s_{\varepsilon_1})$.

As we saw in the Remark on page 89, for each $w \in W$ we may write $\eta_w = \epsilon_w w$ where $\epsilon_w \in T_{\mathbf{Z}}$ is a diagonal matrix with entries $\pm 1$. Therefore, for any $\eta_w,\ \eta_{\bar{w}} \in \mathfrak{M}$ such that $l(w\bar{w}) \neq l(w) + l(\bar{w})$, we have

$$\begin{aligned}
\eta_w\ \eta_{\bar{w}} &= \epsilon_w w\ \epsilon_{\bar{w}} \bar{w} = \epsilon_w^w \epsilon_{\bar{w}}\ .\ w\bar{w} \\
&= \epsilon_w^w \epsilon_{\bar{w}} \epsilon_{w\bar{w}}\ .\ \epsilon_{w\bar{w}}(w\bar{w}) = \epsilon_w^w \epsilon_{\bar{w}} \epsilon_{w\bar{w}}\ .\ \eta_{w\bar{w}} \\
&= \mathcal{I}_{w\bar{w}}\ .\ \eta_{w\bar{w}}.
\end{aligned}$$

where $\mathcal{I}_{w\bar{w}} \in T_{\mathbf{Z}}$ is some diagonal matrix with entries $\pm 1$. So, by using the cocycle rule, for the final term in (7.11) we now have,

$$\begin{aligned}
\sigma_n(\eta_{w_1} w_{\varepsilon_1}, \eta_{w_3}) &= \sigma_n(\mathcal{I}_w \eta_w, \eta_{w_3}), \qquad \text{where } \ w := w_1 s_{\varepsilon_1} \\
&= \frac{\sigma_n(\mathcal{I}_w, \eta_w \eta_{w_3}) \sigma_n(\eta_w, \eta_{w_3})}{\sigma_n(\mathcal{I}_w, \eta_w)} \\
&= \frac{\sigma_n(\mathcal{I}_w, \mathcal{I}_{ww_3} \eta_{ww_3}) \sigma_n(\eta_w, \eta_{w_3})}{\sigma_n(\mathcal{I}_w, \eta_w)} \\
&= \frac{\sigma_n(\mathcal{I}_w, \mathcal{I}_{ww_3}) \sigma_n(\eta_w, \eta_{w_3})}{\sigma_n(\mathcal{I}_w, \eta_w)}, \quad \text{by Corollary 6.} \qquad (7.12)
\end{aligned}$$

However, since we have

$$\begin{aligned}
\sigma_n(\mathcal{I}_w, \mathcal{I}_{ww_3}) &= 1 \quad \text{by Lemma 6,} \\
\sigma_n(\mathcal{I}_w, \eta_w) &= 1 \quad \text{by equation } (i)\text{, Lemma 6,} \\
\sigma_n(\eta_w, \eta_{w_3}) &= 1 \quad \text{by our inductive hypothesis as } l(w_3) \leq \ell_2 - 1,
\end{aligned}$$

equation (7.12) is indeed trivial, Therefore we have been able to prove that, for any $\eta_{w_1},\ \eta_{w_2} \in \mathfrak{M}$, the cocycle $\sigma_n$ satisfies

$$\sigma_n(\eta_{w_1}, \eta_{w_2}) = 1, \quad \text{as required.}$$

□

**Corollary 9** *For each* $\alpha_n$, $\beta_n \in T \subset GL_n(k_\nu)$ *and each* $\eta_{w_1}$, $\eta_{w_2} \in \mathfrak{M}$ *the cocycle* $\sigma_n$ *must satisfy,*

$$\begin{aligned}\sigma_n(\alpha_n\eta_{w_1},\beta_n\eta_{w_2}) &= \sigma_n(\alpha_n\eta_{w_1}\beta_n,\eta_{w_2}).\sigma_n(\alpha_n\eta_{w_1},\beta_n)\\ &= \sigma_n(\alpha_n\ {}^{w_1}\beta_n\eta_{w_1},\eta_{w_2}).\sigma_n(\alpha_n,\eta_{w_1}\beta_n)\sigma_n(\eta_{w_1},\beta_n)\\ &= \sigma_n(\alpha_n\ {}^{w_1}\beta_n,\eta_{w_1}\eta_{w_2}).\sigma_n(\alpha_n,{}^{w_1}\beta_n)\sigma_n(\eta_{w_1},\beta_n).\end{aligned}$$

*Furthermore, whenever we have*

$$l(w_1w_2) = l(w_1) + l(w_2),$$

*our previous results enable us to write,*

$$\begin{aligned}\sigma_n(\alpha_n\eta_{w_1},\beta_n\eta_{w_2}) &= \sigma_n(\alpha_n\ {}^{w_1}\beta_n,\eta_{w_1w_2}).\sigma_n(\alpha_n,{}^{w_1}\beta_n)\sigma_n(\eta_{w_1},\beta_n).\\ &= \sigma_n(\alpha_n,{}^{w_1}\beta_n)\sigma_n(\eta_{w_1},\beta_n)\\ &= \prod_{i<j}(-1)^{\frac{(\rho-1)}{2^r}X_iY_{w_1(j)}}(a_i^{-Y_{w_1(j)}}b_{w_1(j)}^{X_i},\pi)_{\nu,m}\\ &\qquad \prod_{\varsigma\in\Phi(w_1)}(-1)^{\frac{(\rho-1)}{2^r}Y_i}(-1)^{\frac{(\rho-1)}{2^r}Y_iY_j}(b_i^{Y_j}b_j^{-Y_i},\pi)_{\nu,m}.\end{aligned}$$

Let us recall that we had defined $\mathrm{M}_{\mathbf{Z}} \subset \mathrm{M}$ to be the subgroup of monomials generated by the elements,

$$\{w_\varsigma : \varsigma = (i, i+1) \in \Delta\}.$$

Now, using the previous corollary we are further able to deduce the following:

**Corollary 10** *For each* $\mathrm{w}_1$, $\mathrm{w}_2 \in \mathrm{M}_{\mathbf{Z}}$, *we find that the cocycle* $\sigma_n$ *satisfies,*

$$\sigma_n(\mathrm{w}_1, \mathrm{w}_2) = 1.$$

**PROOF:**

Considering the way in which we have defined the set $\mathfrak{M}$ it should be clear that, for each $\mathrm{w}_1, \mathrm{w}_2 \in \mathrm{M}_{\mathbf{Z}}$, we have

$$\mathrm{w}_1 = w_{\varsigma_1}\dots w_{\varsigma_{r_1}} = \mathcal{I}_t\eta_{w_1} \qquad \text{and} \qquad \mathrm{w}_2 = w_{\varsigma_1}\dots w_{\varsigma_{r_2}} = \mathcal{I}_2\eta_{w_2},$$

where, for $\lambda = 1,2$, we have $w_\lambda = s_{\varsigma_1}\dots s_{\varsigma_{r_\lambda}} \in W$ and $\mathcal{I}_\lambda \in T_{\mathbf{Z}}$ is trivial whenever $l(w_\lambda) = r_\lambda$.

Now, using the previous corollary together with Lemmas 5 and 6, we find

$$\begin{aligned}\sigma_n(\mathcal{I}_1\eta_{w_1},\mathcal{I}_2\eta_{w_2}) &= \sigma_n(\mathcal{I}_1\ {}^{w_1}\mathcal{I}_2,\eta_{w_1}\eta_{w_2}).\sigma_n(\mathcal{I}_1,{}^{w_1}\mathcal{I}_2)\sigma_n(\eta_{w_1},\mathcal{I}_2)\\ &= \sigma_n(\mathcal{I}_1\ {}^{w_1}\mathcal{I}_2,\eta_{w_1}\eta_{w_2}).\end{aligned} \tag{7.13}$$

However, as we saw in the proof of Corollary 8, we have

$$\eta_{w_1}\eta_{w_2} =: \mathcal{I}_{w_1w_2}\eta_{w_1w_2} = \mathcal{I}_3\eta_{w_3},$$

where $w_3 = w_1w_2$ and $\mathcal{I}_3$ is trivial whenever $l(w_1w_2) = l(w_1) + l(w_2)$.

Therefore, by letting $\mathcal{I} = \mathcal{I}_1\ {}^{w_1}\mathcal{I}_2 \in T_{\mathbf{Z}}$ and once again using the cocycle rule, equation (7.13) becomes

$$\begin{aligned}\sigma_n(\mathrm{w}_1,\mathrm{w}_2) &= \sigma_n(\mathcal{I}_1\eta_{w_1},\mathcal{I}_2\eta_{w_2})\\ &= \sigma_n(\mathcal{I},\mathcal{I}_3\eta_{w_3})\\ &= \frac{\sigma_n(\mathcal{I}\mathcal{I}_3,\eta_{w_3})\sigma_n(\mathcal{I},\mathcal{I}_3)}{\sigma_n(\mathcal{I}_3,\eta_{w_3})} = 1,\end{aligned}$$

by Lemmas 5 and 6. □

## 7.4 Kubota's Cocycle $\sigma_k$.

It has been shown by Banks, Levy and Sepanski [1] that $\sigma_n$ is the unique cocycle satisfying the relations given in Theorem 1. Then by verifying that $\sigma_k$, the cocycle previously discovered by T. Kubota [5] [6], also satisfies these relations they were able to show that $\sigma_k = \sigma_2$.

Let us recall that Kubota has defined his cocycle $\sigma_k$ by ,

$$\sigma_k(g_1,g_2) = \left(\frac{\chi(g_1g_2)}{\chi(g_1)},\frac{\chi(g_1g_2)}{\chi(g_2)\det(g_1)}\right)^{-1}_{\nu,m},$$

where,

$$\chi(g) = \chi\left(\begin{pmatrix} a & b\\ c & d\end{pmatrix}\right) = \begin{cases} c & \iff c\neq 0\\ d & \iff c = 0\end{cases}$$

for all $g_1$, $g_2 \in \mathrm{SL}_2(k_\nu)$.

We shall now confirm that $\sigma_k = \sigma_2$ on $\mathrm{SL}_2$ by verifying the relations for $\sigma_k$.

Using the definition of the function $\chi$ as given above, for each $\alpha_2, \beta_2 \in T$ in $\mathrm{SL}_2(k_\nu)$ and for $\eta_w \in \mathfrak{M}$, we are able to show that the cocycle $\sigma_k$ satisfies,

$$\begin{aligned}
\sigma_k(\alpha_2, \beta_2) &= (\chi(\alpha_2.\beta_2)/\chi(\alpha_2), \chi(\alpha_2.\beta_2)/\chi(\beta_2)\det(\alpha_2))_{\nu,m}^{-1} \\
&= (\pi^{X_2+Y_2}a_2b_2/\pi^{X_2}a_2, \pi^{X_2+Y_2}a_2b_2/\pi^{Y_2}b_2\pi^{X_1+X_2}a_1a_2)_{\nu,m}^{-1} \\
&= (\pi^{Y_2}b_2, \pi^{-X_1}a_1^{-1})_{\nu,m}^{-1} \\
&= (-1)^{\frac{(\rho-1)}{2^r}X_1Y_1}\ (a_1^{-Y_2}b_2^{X_1}, \pi)_{\nu,m}. \\
\sigma_k(\eta_w, \eta_w) &= (\chi(-I_2)/\chi(\eta_w), \chi(-I_2)/\chi(\eta_w))_{\nu,m} = 1 \\
\\
\sigma_k(\alpha_2, \eta_w) &= (\chi(\alpha_2.\eta_w)/\chi(\alpha_2), \chi(\alpha_2.\eta_w)/\chi(\eta_w)\det(\alpha_2))_{\nu,m}^{-1} \\
&= (\pi^{X_2}a_2/\pi^{X_2}a_2, \pi^{X_2}a_2/\pi^{X_1+X_2}a_1a_2)_{\nu,m}^{-1} \\
&= (1, \pi^{-X_1}a_1^{-1})_{\nu,m}^{-1} = 1, \\
\\
\sigma_k(\eta_w, \alpha_2) &= (\chi(\eta_w.\alpha_2)/\chi(\eta_w), \chi(\eta_w.\alpha_2)/\chi(\alpha_2))_{\nu,m}^{-1} \\
&= (\pi^{X_1}a_1/1, \pi^{X_1}a_1/\pi^{X_2}a_2)_{\nu,m}^{-1} \\
&= (\pi^{X_1}a_1, \pi^{X_1-X_2}a_1a_2^{-1})_{\nu,m}^{-1} \\
&= (-1)^{\frac{(\rho-1)}{2^r}X_1}(-1)^{\frac{(\rho-1)}{2^r}X_1X_2}(a_1^{X_2}a_2^{-X_1}, \pi)_{\nu,m} \\
\\
\sigma_k(\alpha_2\eta_w, \beta_2\eta_w) &= (\chi(\alpha_2\eta_w.\beta_2\eta_w)/\chi(\alpha_2\eta_w), \chi(\eta_w\alpha_2\eta_w.\beta_2\eta_w)/\chi(\beta_2\eta_w)\det(\alpha_2\eta_w))_{\nu,m}^{-1} \\
&= (-\pi^{X_2+Y_1}a_2b_1/\pi^{X_2}a_2, -\pi^{X_2+Y_1}a_2b_1/\pi^{Y_2}b_2\pi^{X_1+X_2}a_1a_2)_{\nu,m}^{-1} \\
&= (-\pi^{Y_1}b_1, -\pi^{-X_1+Y_1-Y_2}a_1^{-1}b_1b_2^{-1})_{\nu,m}^{-1} \\
&= (-1)^{\frac{(\rho-1)}{2^r}(X_1+Y_2)}.(-1)^{\frac{(\rho-1)}{2^r}Y_1+Y_1Y_2+Y_1X_1}(a_1^{-Y_1}b_1^{X_1}, \pi)_{\nu,m}(b_1^{Y_2}b_2^{-Y_1}, \pi)_{\nu,m} \\
&= \sigma_k(\alpha_2\ {}^w\beta_2, \eta_w\eta_w).\sigma_k(\alpha_2, {}^w\beta_2)\sigma_k(\eta_w, \beta_2), \quad \text{as required.}
\end{aligned}$$

By simply considering the function $\chi$ given in the definition of Kubota's cocycle and calculating for general matrices $n_1, n_2 \in N, g_1, g_2 \in \mathrm{SL}_2(k_\nu)$ we also find that the cocycle $\sigma_k$ satisfies,

$$\begin{aligned}
\sigma_k(n_1, g_1) = \sigma_k(g_1, n_1) &= 1 \\
\sigma_k(n_1g_1, g_2n_2) &= \sigma_k(g_1, g_2) \\
\sigma_k(g_1n_1, g_2) &= \sigma_k(g_1, n_1g_2).
\end{aligned}$$

Finally, by simply checking these formulae with those given in Lemmas 5, 6 and 7 we do indeed find that the cocycle $\sigma_2$, when restricted to $\mathrm{SL}_2(k_\nu)$, is precisely the cocycle $\sigma_k$ first discovered by Kubota.

# Chapter 8

# The splitting of the quotient $\frac{dec_\nu}{\sigma_n}$

## 8.1 Introduction

Let $k_\nu$ be a local field with valuation $\nu$ and a fixed uniformizing element $\pi \in \mathfrak{O}_\nu$.

In the previous chapters we have considered two distinct cocycles $dec_\nu$ and $\sigma_n$ in $Z^2(\mathrm{GL}_n(k_\nu), \mu_m)$ corresponding to metaplectic covers of $\mathrm{GL}_n(k_\nu)$. Now, as we saw in Section 1.1.1, these two cocycles give us two realisations of the local metaplectic cover,

$$1 \to \mu_m \longrightarrow \begin{matrix} \tilde{\mathcal{G}}_{dec_\nu} \\ \tilde{\mathcal{G}}_{\sigma_n} \end{matrix} \longrightarrow \mathrm{GL}_n(k_\nu) \to 1.$$

By realising each of these metaplectic groups as the set of pairs $\mathrm{GL}_n(k_\nu) \times \mu_m$, such that,

$$\begin{aligned} \tilde{\mathcal{G}}_{dec_\nu} &= \langle (g, \xi) : (g, \xi).(g', \xi') = (gg', \xi\xi' \, dec_\nu(g, g')) \rangle \\ \tilde{\mathcal{G}}_{\sigma_n} &= \langle (g, \xi) : (g, \xi).(g', \xi') = (gg', \xi\xi' \sigma_n(g, g')) \rangle, \end{aligned}$$

we have,

$$\tilde{\mathcal{G}}_{\sigma_n} \cong |_{\mathrm{SL}_n} \tilde{\mathcal{G}}_{dec_\nu}, \quad \text{given by,} \;\; (g, \xi) \longmapsto (g, \xi\psi(g)),$$

where the map $\psi$ clearly satisfies,

$$\delta\psi = \frac{dec_\nu}{\sigma_n}.$$

Therefore, by studying the splitting of the quotient of the cocycles $\sigma_n$ and $dec_\nu$ on $\mathrm{SL}_n$, in this chapter we shall be able to calculate explicit formulae for the function $\psi$ and thus find this isomorphism.

Furthermore, by slightly changing the definition of the cocycle $\sigma_n$ given in Section 7.2, on page 187, we are able to show that when the number of roots of unity in $\mu_m$ is odd the isomorphism given above extends to the whole of $\mathrm{GL}_n(k_\nu)$. When the number of roots of unity in $\mu_m$ is even this is no longer the case.

### 8.1.1 Commutators

If $\vartheta \in Z^2(G, \mu_m)$ is any 2-cocycle corresponding to the central extension,

$$1 \longmapsto \mu_m \longmapsto \tilde{G} \longmapsto G \longmapsto 1,$$

then, for any abelian subgroup $T \subseteq G$, we define the commutator of $\vartheta$ to be the function,

$$\begin{aligned} & [-,-]_\vartheta \;:\; T \times T \longmapsto \mu_m \\ \text{such that} \qquad & [t_1, t_2]_\vartheta = \frac{\vartheta(t_1, t_2)}{\vartheta(t_2, t_1)}, \qquad t_1, t_2 \in T. \end{aligned}$$

It is easily shown that $[-,-]_\vartheta$ is bilinear, skew symmetric and depends only on $[\vartheta]$, the cohomology class of $\vartheta$.

Let us now suppose that the cocycle $\vartheta$ splits on the subgroup $T \subseteq G$. Then, there exists some cochain $\kappa$ such that $\vartheta = \partial\kappa$. Therefore, for each $t_1, t_2 \in T$, we find

$$\begin{aligned} [t_1, t_2]_\vartheta = \frac{\vartheta(t_1, t_2)}{\vartheta(t_2, t_1)} &= \frac{\kappa(t_1)\kappa(t_2)}{\kappa(t_1.t_2)} . \frac{\kappa(t_2.t_1)}{\kappa(t_2)\kappa(t_1)} \\ &= \frac{\kappa(t_2.t_1)}{\kappa(t_1.t_2)} = 1, \quad \text{since } T \subseteq G \text{ is abelian.} \end{aligned}$$

Therefore any cocycle which splits on an abelian subgroup $T \subseteq G$ will have a trivial commutator on $T$.

### 8.1.2 Homomorphisms

#### The local field $k_\nu$

Since $m$ and $\pi$ have been taken to be coprime we have,

$$k_\nu{}^\times/(k_\nu{}^\times)^m \;\cong\; \mathbb{Z}/m \oplus \mu_m, \qquad \text{given by,} \qquad \pi^X a \longmapsto (X \pmod m), (a, \pi)_{\nu,m}).$$

Therefore, when considering the set $\mathrm{Hom}(k_\nu{}^\times, \mu_m)$, we find

$$\mathrm{Hom}(k_\nu{}^\times, \mu_m) \cong \mathrm{Hom}(k_\nu{}^\times/(k_\nu{}^\times)^m, \mu_m) \cong \mathrm{Hom}(\mathbb{Z}/m \oplus \mu_m, \mu_m) \cong \mu_m \oplus \mathbb{Z}/m,$$

and any homomorphism $k_\nu{}^\times \mapsto \mu_m$ must be of the form,

$$\pi^X a \longmapsto \xi^X (a, \pi)^\Lambda_{\nu,m},$$

for suitable $\xi \in \mu_m$ and $\Lambda \in \mathbb{Z}/m$.

### Torus

Let us now consider the set $\mathrm{Hom}(T, \mu_m)$ where $T$ is the torus of diagonal matrices in $\mathrm{GL}_n(k_\nu)$. Clearly, since we have $T \cong k_\nu{}^\times \times \ldots \times k_\nu{}^\times = \oplus_1^n k_\nu{}^\times$, we must have

$$\mathrm{Hom}(T, \mu_m) \cong \mathrm{Hom}(\oplus_1^n k_\nu{}^\times, \mu_m) \cong \mathrm{Hom}(k_\nu{}^\times, \mu_m)^n.$$

Therefore it follows that any homomorphism $T \mapsto \mu_m$ must be of the form,

$$\alpha_n = \mathrm{diag}(\pi^{X_1} a_1, \ldots, \pi^{X_n} a_n) \longmapsto \xi_1^{X_1}(a_1, \pi)_{\nu,m}^{\Lambda_1} \ldots \xi_n^{X_n}(a_n, \pi)_{\nu,m}^{\Lambda_n},$$

for suitable $\xi_1, \ldots, \xi_n \in \mu_m$ and $\Lambda_1, \ldots, \Lambda_n \in \mathbb{Z}/m$.

### Monomials

In this section we shall consider the set of homomorphisms from $M$, the monomials in $\mathrm{GL}_n(k_\nu)$, to the group $\mu_m$ of roots of unity. Here we shall consider the two cases when $m$ is odd and $m$ is even separately.

**Suppose $m$ is odd:**

In this case it can be shown that,

$$M/M' \cong k_\nu{}^\times \oplus S_n/A_n, \quad \text{given by, } m = \alpha_n \eta_w \longmapsto (\det(\alpha_n), \mathrm{sign}(w)).$$

Therefore, since $\mathrm{sign}(w) = \pm 1$, in this case we find that

$$\mathrm{Hom}(M, \mu_m) \cong \mathrm{Hom}(M/M', \mu_m) \cong \mathrm{Hom}(k_\nu{}^\times, \mu_m).$$

So, when $m$ is odd, any homomorphism $M \mapsto \mu_m$ must be of the form,

$$m = \alpha_n \eta_w \longmapsto \xi^{X_1 + \ldots + X_n}(a_1 \ldots a_n, \pi)_{\nu,m}^{\Lambda},$$

for suitable $\xi \in \mu_m$ and $\Lambda \in \mathbb{Z}/m$ and where $\det(\alpha_n) = \pi^{X_1} a_1 \ldots \pi^{X_n} a_n$.

**Suppose $m$ is even:**

In the work which follows we shall see that when $m$ is even we need only consider the quotient of the cocycles $dec_\nu$ and $\sigma_n$ on $\mathrm{SL}_n(k_\nu)$. Therefore, in this section, we shall consider homomorphisms from the monomials $\mathbb{M} \subset \mathrm{SL}_n(k_\nu)$ to the group $\mu_{2^r}$.

- $n = 2:$ In this case it can be shown that,

$$\mathbb{M}/\mathbb{M}' = \mathbb{M}/\langle \begin{pmatrix} a^2 & 0 \\ 0 & a^{-2} \end{pmatrix} \rangle \cong k_\nu{}^\times/(k_\nu{}^\times)^2 \oplus S_2/A_2$$

$$\Rightarrow \quad \mathrm{Hom}(\mathbb{M}, \mu_m) \cong \mathrm{Hom}(k_\nu{}^\times/(k_\nu{}^\times)^2 \oplus C_2, \mu_m),$$

and any homomorphism $\mathbb{M} \mapsto \mu_m$ must be of the form,

$$m = \begin{pmatrix} \pi^{X} a & 0 \\ 0 & \pi^{-X} a^{-1} \end{pmatrix} \eta_w \longmapsto (-1)^{X\Lambda_1} \, (a,\pi)_{\nu,2}^{\Lambda_2} \, \mathrm{sign}(w)^{\Lambda_3},$$

for some $\Lambda_1, \Lambda_2, \Lambda_3 \in \{0,1\}$.

- $n \geq 3$:

Here we simply find that,

$$\begin{aligned} & \mathbb{M}/\mathbb{M}' \cong S_n/A_n, \quad \text{given by, } m = \alpha_n \eta_w \longmapsto \mathrm{sign}(w) \\ \Rightarrow \quad & \mathrm{Hom}(\mathbb{M}, \mu_m) \cong \mathrm{Hom}(\mathbb{M}/\mathbb{M}', \mu_m) \cong \mathrm{Hom}(S_n/A_n, \mu_m). \end{aligned}$$

Therefore, when $n \geq 3$, any homomorphism $\mathbb{M} \mapsto \mu_m$ is found to satisfy,

$$m = \alpha_n \eta_w \longmapsto \mathrm{sign}(w)^{\Lambda}, \qquad \text{for some } \Lambda = 0, 1.$$

**The group $N$**

As we have seen before, when $n = 2$ we find that $N_2 = \langle \left( \begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix} \right) \rangle \in \mathrm{SL}_2(k_\nu)$ is abelian. Therefore we have,

$$\mathrm{Hom}(N_2, \mu_m) \cong \mathrm{Hom}(k_\nu, \mu_m) \cong \mathrm{Hom}(k_\nu / m\, k_\nu, \mu_m) = 1.$$

Now, it can also be shown that when $n \geq 3$ we have,

$$N_n' = \left\langle \begin{pmatrix} 1 & 0 & * \\ & \cdot & 0 \\ 0 & & 1 \end{pmatrix} \right\rangle \in \mathrm{SL}_n(k_\nu).$$

Therefore we find that,

$$\begin{aligned} \mathrm{Hom}(N_n, \mu_m) &\cong \mathrm{Hom}(N_n/N_n', \mu_m) \\ &\cong \mathrm{Hom}(\oplus_1^{n-1} k_\nu, \mu_m) \cong \mathrm{Hom}(k_\nu / m\, k_\nu, \mu_m)^{n-1} = 1. \end{aligned}$$

So, in conclusion we have found that, for any $n$ and with $m$ odd or even, the group $N$ satisfies,

$$\mathrm{Hom}(N, \mu_m) = 1.$$

**The groups $\mathrm{GL}_n(k_\nu)$ and $\mathrm{SL}_n(k_\nu)$**

For $\mathrm{GL}_n(k_\nu)$ we find,

$$\begin{aligned} \mathrm{Hom}(\mathrm{GL}_n(k_\nu), \mu_m) &\cong \mathrm{Hom}(\mathrm{GL}_n(k_\nu)/\mathrm{GL}_n(k_\nu)', \mu_m) \\ &\cong \mathrm{Hom}(\mathrm{GL}_n(k_\nu)/\mathrm{SL}_n(k_\nu), \mu_m) \cong \mathrm{Hom}({k_\nu}^\times, \mu_m). \end{aligned}$$

Therefore, once again we find that any homomorphism $\mathrm{GL}_n(k_\nu) \mapsto \mu_m$ must be of the form,

$$\gamma \longmapsto \xi^{\mathcal{G}}(g,\pi)^{\Lambda}_{\nu,m},$$

for some $\xi \in \mu_m$ and $\Lambda \in \mathbb{Z}/m$ and where $\det(\gamma) = \pi^{\mathcal{G}} g$.

For $\mathrm{SL}_n(k_\nu)$ we simply find,

$$\mathrm{Hom}(\mathrm{SL}_n(k_\nu), \mu_m) = 1.$$

## 8.2 Defining the cocycle $\mathrm{Dec}^{\sigma}_{\nu}$

Let us begin by considering the quotient of the two cocycles $dec_\nu$ and $\sigma_n$ when restricted to the abelian subgroup $T \subset \mathrm{GL}_n(k_\nu)$ of diagonal matrices. Using Theorem 2.4.1 and Lemma 5, for each $\alpha_n,\ \beta_n \in T$, we have

$$\begin{aligned}
\frac{dec_\nu}{\sigma_n}(\alpha_n,\beta_n) &= \frac{\prod_{i=1}^n (a_i^{Y_i},\pi)_{\nu,m} \prod_{i<j} (-1)^{\frac{(\rho-1)}{2^r} Y_i X_j} \,.\, \partial\tau_n(\alpha_n,\beta_n)}{\prod_{i<j} (-1)^{\frac{(\rho-1)}{2^r} X_i Y_j} (a_i^{-Y_j} b_j^{X_i},\pi)_{\nu,m}} \\
&= \prod_{i<j} (-1)^{\frac{(\rho-1)}{2^r}(X_iY_j+Y_iX_j)} \prod_{i<j} (a_i^{Y_j} b_j^{-X_i},\pi)_{\nu,m} \prod_{i=1}^{n} (a_i^{Y_i},\pi)_{\nu,m} \,.\, \partial\tau_n(\alpha_n,\beta_n) \\
&= \prod_{i<j} (-1)^{\frac{(\rho-1)}{2^r}(X_iY_j+Y_iX_j)} \prod_{i\le j} (a_i^{Y_j},\pi)_{\nu,m} \prod_{i<j} (b_j^{-X_i},\pi)_{\nu,m} \,.\, \partial\tau_n(\alpha_n,\beta_n).
\end{aligned}$$

Calculating the commutator of the quotient $dec_\nu/\sigma_n$ on the abelian subgroup $T \subset \mathrm{GL}_n(k_\nu)$ we now find,

$$\begin{aligned}
[\,\alpha_n,\beta_n\,]_{\frac{dec_\nu}{\sigma_n}} &= \frac{\prod_{i<j}(-1)^{\frac{(\rho-1)}{2^r}(X_iY_j+Y_iX_j)} \prod_{i\le j}(a_i^{Y_j},\pi)_{\nu,m} \prod_{i<j}^{n}(b_j^{-X_i},\pi)_{\nu,m}}{\prod_{i<j}(-1)^{\frac{(\rho-1)}{2^r}(Y_iX_j+X_iY_j)} \prod_{i\le j}(b_i^{X_j},\pi)_{\nu,m} \prod_{i<j}^{n}(a_j^{-Y_i},\pi)_{\nu,m}} \\
&= \prod_{i,j} (a_i^{Y_j} b_i^{-X_j},\pi)_{\nu,m} \\
&= (\det(\alpha_n),\det(\beta_n))_{\nu,m}.
\end{aligned} \tag{8.1}$$

Therefore, if we restrict to $\mathrm{SL}_n(k_\nu)$, we do indeed find that this commutator is trivial.

Let us now consider the case when $m$, the number of roots of unity in $\mu_m$, is odd. In this case we notice that we may write,

$$(\det(\alpha_n),\det(\beta_n))_{\nu,m} = \frac{(\det(\alpha_n),\det(\beta_n))^{1/2}_{\nu,m}}{(\det(\alpha_n),\det(\beta_n))^{-1/2}_{\nu,m}}, \tag{8.2}$$

which gives us the idea to return to the definition of the cocycle $\sigma_n \in Z^2(\mathrm{GL}_n(k_\nu), \mu_m)$ given in Section 7.2 on page 187. There we had defined,

$$\sigma_n(g_1, g_2) = \sigma_{SL_{n+1}}(\imath(g_1), \imath(g_2)).(\det(g_1), \det(g_2))_{\nu,m}^{-1}.$$

Considering equation (8.2) we now define a new 2-cocycle $\sigma_n^*$ by,

$$\begin{aligned}\sigma_n^*(g_1, g_2) &= \sigma_{SL_{n+1}}(\imath(g_1), \imath(g_2)).(\det(g_1), \det(g_2))_{\nu,m}^{-1/2} \\ &= \sigma_n(g_1, g_2).(\det(g_1), \det(g_2))_{\nu,m}^{1/2}.\end{aligned}$$

Then, when $m$ is odd, the cocycle $\sigma_n^* \in Z^2(\mathrm{GL}_n(k_\nu), \mu_m)$ is well defined and corresponds to the metaplectic extension,

$$1 \longrightarrow \mu_m \longrightarrow \tilde{\mathcal{G}}_{\sigma_n^*} \longrightarrow \mathrm{GL}_n(k_\nu) \longrightarrow 1.$$

If we restrict to $\mathrm{SL}_n(k_\nu)$ it should be clear that this extension is identical to the extension corresponding to $\sigma_n$.

Whenever $m$ is even there is an obvious problem with this definition. That is, we shall have no way of making sense of the term $(\det(g_1), \det(g_2))_{\nu,m}^{1/2}$. In order to get round this, whenever we are dealing with the case that $m$ is even, we shall restrict to $\mathrm{SL}_n(k_\nu)$. Then we simply have,

$$\sigma_n^*(g_1, g_2)|_{\mathrm{SL}_n(k_\nu)} = \sigma_n(g_1, g_2),$$

is again well defined.

**Remark:**

So far in this thesis we have been able to consider the cases when $m$ is odd and even simultaneously. Since we would like to continue doing so, for the remainder of this thesis, we shall restrict our attention to $\mathrm{SL}_n(k_\nu)$ whenever $m = 2^r$ is even. With this in mind, let us now define the notation,

$$G_\nu = \begin{cases} \mathrm{GL}_n(k_\nu) & \text{whenever } m \text{ is odd.} \\ \mathrm{SL}_n(k_\nu) & \text{whenever } m \text{ is even.} \end{cases}$$

**Definition:**

We define the cocycle $\mathrm{Dec}_\nu^\sigma$ on $G_\nu$ with values in $\mu_m$ by,

$$\mathrm{Dec}_\nu^\sigma(g_1, g_2) = \frac{dec_\nu}{\sigma_n^*}(g_1, g_2) = \frac{dec_\nu(g_1, g_2)}{\sigma_n^*(g_1, g_2)}, \tag{8.3}$$

for all $g_1,\ g_2 \in G_\nu$.

As stated in the previous remark, from this point on it is understood that whenever $m$ is even we shall only consider the cocycle $\mathrm{Dec}_\nu^\sigma$ on $\mathrm{SL}_n(k_\nu)$. Then any terms involving determinants will be trivial and we shall have,

$$\mathrm{Dec}_\nu^\sigma(g_1, g_2)|_{\mathrm{SL}_n} = \frac{dec_\nu}{\sigma_n}(g_1, g_2),$$

for all $g_1,\ g_2 \in \mathrm{SL}_n(k_\nu) = G_\nu$.

For the remainder of this chapter we shall concern ourselves with finding the splitting of this cocycle $\mathrm{Dec}_\nu^\sigma$ on $G_\nu$. That is, we shall find the function $\psi$ satisfying,

$$\mathrm{Dec}_\nu^\sigma(g_1, g_2) = \frac{\psi(g_1)\psi(g_2)}{\psi(g_1 g_2)}, \qquad \forall\, g_1,\ g_2 \in G_\nu$$

and corresponding to the isomorphisms,

- when $m$ is odd: $\tilde{\mathcal{G}}_{\sigma_n^*} \cong |_{\mathrm{GL_n}}\tilde{\mathcal{G}}_{dec_\nu}$
- when $m$ is even: $\tilde{\mathcal{G}}_{\sigma_n^*} = \tilde{\mathcal{G}}_{\sigma_n} \cong |_{\mathrm{SL_n}}\tilde{\mathcal{G}}_{dec_\nu},$

given by, $(g, \xi) \longmapsto (g, \xi\psi(g))$.

## 8.3 The splitting of $\mathrm{Dec}_\nu^\sigma$ on the Torus in $G_\nu$

In this section we shall find all possible splittings of the cocycle $\mathrm{Dec}_\nu^\sigma$ on the torus $T$ of diagonal matrices.

**Theorem 8.3.1** *The cocycle* $\mathrm{Dec}_\nu^\sigma$ *splits on the torus* $T \subset G_\nu$ *over* $\mu_m$*. A cochain which spits the cocycle* $\mathrm{Dec}_\nu^\sigma$ *on* $T$ *is given by,*

$$\psi_T(\alpha_n) = \tau_n(\alpha_n) \cdot \prod_{\varsigma\in\Phi^+} (-1)^{\frac{(\rho-1)}{2^r}X_iX_j}(a_j^{X_i}, \pi)_{\nu,m} \cdot \prod_{i=1}^{n}(a_i, \pi)_{\nu,m}^{-\sum_{j=1}^{n} X_j/2}, \tag{8.4}$$

*where* $\tau_n$ *is the function described in Chapter 3.*

Let us be clear that, in equation (8.4), when $m$ is odd it is understood that $(-1) = 1$ and when $m$ is even it is understood that $\det(\alpha_n) = 1$.

**PROOF OF THEOREM:**

By simply calculating, for each $\alpha_n,\ \beta_n \in T \subset G_\nu$, we find that the cocycle $\mathrm{Dec}_\nu^\sigma$ satisfies,

$$\mathrm{Dec}_\nu^\sigma(\alpha_n,\beta_n) = \frac{dec_\nu(\alpha_n,\beta_n)}{\sigma_n^*(\alpha_n,\beta_n)} = \frac{dec_\nu(\alpha_n,\beta_n)}{\sigma_n(\alpha_n,\beta_n)}.(\det(\alpha_n),\det(\beta_n))_{\nu,m}^{-1/2}$$

$$= \partial\tau_n(\alpha_n,\beta_n).\prod_{i<j}(-1)^{\frac{(\rho-1)}{2^r}(X_iY_j+Y_iX_j)}\prod_{i\leq j}(a_i^{Y_j},\pi)_{\nu,m}\prod_{i<j}(b_j^{-X_i},\pi)_{\nu,m}.$$
$$.\prod_{i=1}^{n}(b_i^{\sum_{j=1}^n X_j/2},\pi)_{\nu,m}(a_i^{-\sum_{j=1}^n Y_j/2},\pi)_{\nu,m}$$

$$= \partial\tau_n(\alpha_n,\beta_n).\prod_{i<j}(-1)^{\frac{(\rho-1)}{2^r}(X_iY_j+Y_iX_j)}$$
$$.\prod_{i=1}^{n}(a_i^{\sum_{j=i}^n Y_j-\sum_{j=1}^n Y_j/2},\pi)_{\nu,m}.(b_i^{-\sum_{j=1}^{i-1}X_j+\sum_{j=1}^n X_j/2},\pi)_{\nu,m}$$

$$= \partial\tau_n(\alpha_n,\beta_n).\prod_{i\leq j}(-1)^{\frac{(\rho-1)}{2^r}(X_iY_j+Y_iX_j)}$$
$$.\prod_{i=1}^{n}(a_i^{-\sum_{j=1}^{i-1}Y_j+\sum_{j=1}^n Y_j/2},\pi)_{\nu,m}.(b_i^{-\sum_{j=1}^{i-1}X_j+\sum_{j=1}^n X_j/2},\pi)_{\nu,m}$$

$$= \partial\tau_n(\alpha_n,\beta_n).\prod_{\varsigma\in\Phi^+}(-1)^{\frac{(\rho-1)}{2^r}(X_iY_j+Y_iX_j)}$$
$$.\prod_{\varsigma\in\Phi^+}(a_j^{-Y_i},\pi)_{\nu,m}\prod_{i=1}^{n}(a_i^{\sum_{j=1}^n Y_j/2},\pi)_{\nu,m}.\prod_{\varsigma\in\Phi^+}(b_j^{-X_i},\pi)_{\nu,m}\prod_{i=1}^{n}(b_i^{\sum_{j=1}^n X_j/2},\pi)_{\nu,m}$$

$$= \frac{\psi_T(\alpha_n)\psi_T(\beta_n)}{\psi_T(\alpha_n\beta_n)}.$$

So, when $m$ is both odd and even, a cochain which splits $\mathrm{Dec}_\nu^\sigma$ on $T \subset G_\nu$ is given by,

$$\psi_T(\alpha_n) = \tau_n(\alpha_n)\ .\ \prod_{\varsigma\in\Phi^+}(-1)^{\frac{(\rho-1)}{2^r}X_iX_j}(a_j^{X_i},\pi)_{\nu,m}.\prod_{i=1}^{n}(a_i^{-\sum_{j=1}^n X_j/2},\pi)_{\nu,m},$$

which completes the proof of our theorem. □

**Remark:**

Let us be clear that, although we have found a splitting of $\mathrm{Dec}_\nu^\sigma$ on $T$, the cochain $\psi_T$ is not unique. However, considering the fact that,

$$\mathrm{Dec}_\nu^\sigma(\alpha_n,\beta_n) = \frac{\psi_T(\alpha_n)\psi_T(\beta_n)}{\psi_T(\alpha_n\beta_n)},$$

we see that any other splitting on $T$ must be obtained by multiplying $\psi_T$ with some homomorphism $T \longmapsto \mu_m$. As we have seen, in Section 8.1.2, any such homomorphism $\phi$ must be of the form,

$$\phi(\alpha_n) = \xi_1^{X_1}(a_1,\pi)_{\nu,m}^{\Lambda_1}\cdots\xi_n^{X_n}(a_n,\pi)_{\nu,m}^{\Lambda_n},$$

for suitable $\xi_1,\ldots,\xi_n \in \mu_m$ and $\Lambda_1,\ldots,\Lambda_n \in \mathbb{Z}/m$.

**Corollary 11** *Any cochain $\tilde{\psi}_T$ which splits the cocycle* $\mathrm{Dec}_\nu^\sigma$ *on the torus $T \subset G_\nu$ must satisfy,*

$$\tilde{\psi}_T(\alpha_n) = \psi_T(\alpha_n).\phi(\alpha_n),$$

*where $\psi_T$ is as above and $\phi$ is some homomorphism in* $\mathrm{Hom}(T, \mu_m)$.

## 8.4 The splitting of $\mathrm{Dec}_\nu^\sigma$ on the Monomials in $G_\nu$

In this section we shall find all possible splittings of the cocycle $\mathrm{Dec}_\nu^\sigma$ on the monomials. That is, we shall find all functions $\tilde{\psi}_M$ which satisfy,

$$\mathrm{Dec}_\nu^\sigma(\mathrm{m}_1, \mathrm{m}_2) = \frac{\tilde{\psi}_M(\mathrm{m}_1)\tilde{\psi}_M(\mathrm{m}_2)}{\tilde{\psi}_M(\mathrm{m}_1\mathrm{m}_2)},$$

for each $\mathrm{m}_1,\ \mathrm{m}_2 \in M \subset G_\nu$.

By assuming it is understood that $(-1) = 1$ when $m$ is odd and $\det(\alpha_n) = 1$ when $m$ is even we have been able to deal with both cases simultaneously. In order to keep doing so we shall first need to re-define the function sign. That is, from this point on, we define the function $\mathrm{sign} : W \longmapsto \mu_m$ by,

$$\mathrm{sign}(w) = \Big(\prod_{\xi \in \mu_m} \xi\Big)^{l(w)} = (-1)^{l(w)}, \qquad \text{where } (-1) = 1 \text{ when } m \text{ is odd.}$$

This simply allows us to say that the function sign is defined as usual when $m$ is even but is trivial whenever the number of roots of unity in $\mu_m$ is odd.

Let us also be clear that, having found a cochain $\psi_M$ which splits the cocycle $\mathrm{Dec}_\nu^\sigma$ on $M \in G_\nu$, any other possible cochain will be of the form,

$$\tilde{\psi}_M(m) = v(m).\psi_M(m),$$

where $v : M \longmapsto \mu_m$ is some homomorphism satisfying,

$$v(\alpha_n \eta_w) = \begin{cases} \varphi(\det(\alpha_n)) & \text{some } \varphi \in \mathrm{Hom}(k_\nu{}^\times, \mu_m), \text{ when } m \text{ is odd} \\ \mathrm{sign}(w)^\Lambda & \text{some } \Lambda \in \{0, 1\}, \text{ when } m \text{ is even.} \end{cases}$$

Before we move on we shall first require one more result concerning products over root spaces.

**Lemma 8.4.1** *Let us define the function* $\chi(i,j) : T \mapsto \mu_m$ *by,*

$$(1) \qquad \chi(i,j) = (a_i^{X_j}, \pi)_{\nu,m} \qquad\qquad (2) \qquad \chi(i,j) = (-1)^{\frac{(p-1)}{2^r} X_i},$$

*then, in both cases* (1) *and* (2), *we find*

$$\prod_{(i,j)\in\Phi^+} \frac{\chi(\,i\,,\,j\,)}{\chi(w^{-1}(i), w^{-1}(j))} = \prod_{(i,j)\in\Phi(w)} \frac{\chi(i,j)}{\chi(j,i)}, \tag{8.5}$$

**PROOF:**

By simply calculating we do indeed find,

$$\begin{aligned}
\prod_{(i,j)\in\Phi^+} \frac{\chi(\,i\,,\,j\,)}{\chi(w^{-1}(i), w^{-1}(j))} &= \frac{\prod_{(i,j)\in\Phi^+} \quad \chi(\,i\,,\,j\,)}{\prod_{(i,j)\in\Phi^+} \chi(w^{-1}(i), w^{-1}(j))} \\
&= \frac{\prod_{(i,j)\in\Phi^+} \chi(i,j)}{\prod\limits_{(w(i),w(j))\in\Phi^+} \chi(i,j)} \\
&= \frac{\prod_{(i,j)\in\Phi^+} \chi(i,j)}{\prod\limits_{(i,j)\in\Phi^+:(w(i),w(j))\in\Phi^+} \chi(i,j) \prod\limits_{(i,j)\in\Phi^-:(w(i),w(j))\in\Phi^+} \chi(i,j)} \\
&= \frac{\prod_{(i,j)\in\Phi^+} \chi(i,j)}{\prod_{(i,j)\in\Phi^+\setminus\Phi(w)} \chi(i,j) \prod_{(j,i)\in\Phi(w)} \chi(i,j)} \\
&= \frac{\prod_{(i,j)\in\Phi(w)} \chi(i,j)}{\prod_{(j,i)\in\Phi(w)} \chi(i,j)} \\
&= \prod_{(i,j)\in\Phi(w)} \frac{\chi(i,j)}{\chi(j,i)},
\end{aligned}$$

which completes the proof of our lemma.

□

This result shall be important in various calculations throughout the remaining proofs in this section.

**Theorem 8.4.1** *Any cochain* $\tilde{\psi}_M$ *which splits the cocycle* $\mathrm{Dec}_\nu^\sigma$ *over* $M \in G_\nu$ *must satisfy,*

$$\tilde{\psi}_M(\eta_w) = \mathrm{sign}(w)^\Lambda \qquad\qquad \textit{for some} \quad \Lambda \in \{0,1\},$$

*where* sign *is trivial whenever* $m$ *is odd.*

**PROOF OF THEOREM:**

Let us recall that we have defined the set $\mathbb{M}_{\mathbf{Z}} = \langle w_\varsigma : \varsigma \in \Delta \rangle \subset \mathbb{M}$. Then, any matrix $\mathrm{w} \in \mathbb{M}_{\mathbf{Z}}$ may be written as,

$$\mathrm{w} = w_{\varsigma_1} \dots w_{\varsigma_r} = \mathcal{I}\eta_w,$$

where $w = s_{\varsigma_1} \dots s_{\varsigma_r} \in W$ and $\mathcal{I} \in T_{\mathbf{Z}}$ is trivial whenever $l(w) = r$.
Using Theorem 1.4.1 and Corollary 10 we find that,

$$dec_\nu(\mathrm{w}_1, \mathrm{w}_2) = \sigma_n(\mathrm{w}_1, \mathrm{w}_2) = 1.$$

We also know that $\det(\mathrm{w}) = 1$ for each $\mathrm{w} \in \mathbb{M}_{\mathbf{Z}}$. Therefore, for each $\mathrm{w}_1$, $\mathrm{w}_2 \in \mathbb{M}_{\mathbf{Z}}$ we find,

$$\mathrm{Dec}_\nu^\sigma(\mathrm{w}_1, \mathrm{w}_2) = \frac{dec_\nu(\mathrm{w}_1, \mathrm{w}_2)}{\sigma_n(\mathrm{w}_1, \mathrm{w}_2)} = 1,$$

and therefore any cochain which splits the cocycle must satisfy,

$$\tilde{\psi}_M(\mathrm{w}_1\mathrm{w}_2) = \tilde{\psi}_M(\mathrm{w}_1)\tilde{\psi}_M(\mathrm{w}_2).$$

So, when we restrict to the subgroup $\mathbb{M}_{\mathbf{Z}} \subset \mathbb{M}$ we find that any possible cochain $\tilde{\psi}_M$ must also be a homomorphism. Now, since

$$\mathbb{M}_{\mathbf{Z}}/\mathbb{M}'_{\mathbf{Z}} \cong S_n/A_n, \qquad \text{given by,} \qquad \mathrm{w} = \mathcal{I}\eta_w \longmapsto \mathrm{sign}(w)$$

we find that,

$$\mathrm{Hom}(\mathbb{M}_{\mathbf{Z}}, \mu_m) \cong \mathrm{Hom}(S_n/A_n, \mu_m).$$

Therefore, on the subgroup $\mathbb{M}_{\mathbf{Z}}$, any cochain $\tilde{\psi}_M$ must satisfy,

$$\begin{aligned} \tilde{\psi}_M(\mathrm{w}) &= \begin{cases} 1 & \text{whenever } m \text{ is odd.} \\ \mathrm{sign}(w)^\Lambda & \text{whenever } m \text{ is even.} \end{cases} \\ &= \mathrm{sign}(w)^\Lambda \qquad \text{for some } \Lambda \in \{0, 1\}, \end{aligned}$$

for each $\mathrm{w} = \mathcal{I}\eta_w \in \mathbb{M}_{\mathbf{Z}}$ and where the function sign is defined as on page 205.

Finally, since for each $w \in W$ we have $\eta_w \in \mathbb{M}_{\mathbf{Z}}$, our result immediately follows.

□

**Theorem 8.4.2** *Any function $\tilde{\psi}_M$ which splits $\mathrm{Dec}_\nu^\sigma$ on $M \subset G_\nu$ over $\mu_m$ satisfies,*

*(i)* $\tilde{\psi}_M(\alpha_n\eta_w) = \tilde{\psi}_M(\alpha_n)\,\mathrm{sign}(w)^\Lambda, \qquad \Lambda \in \{0, 1\}$

*(ii)* $$\tilde{\psi}_M(\alpha_n) = \tilde{\psi}_M({}^w\alpha_n) \prod_{(i,j)\in\Phi(w)} (-1)^{\frac{(\rho-1)}{2r}(X_i+X_j+X_iX_j)}(-1)^{\frac{(\rho-1)}{2r}\max\{X_i,X_j\}}(a_j^{X_i}/a_i^{X_j}, \pi)_{\nu,m},$$

*for each $\alpha_n \in T$, $\eta_w \in \mathfrak{M}$.*

Once again we are assuming it is understood that $(-1)$ and $\text{sign}(w)$ are trivial whenever $m$ is odd and $\det(\alpha_n) = 1$ when $m$ is even.

**PROOF OF THEOREM:**

We shall begin by considering statement $(i)$. Referring to Theorem 4.6.1 and Lemma 6 we have found that for any $\alpha_n \in T$, $\eta_w \in \mathfrak{M}$ the cocycles $dec_\nu$ and $\sigma_n$ satisfy,

$$dec_\nu(\alpha_n, \eta_w) = \sigma_n(\alpha_n, \eta_w) = 1.$$

Since $\det(\eta_w) = 1$ we are immediately able to deduce that the cocycle $\text{Dec}_\nu^\sigma$ satisfies,

$$\text{Dec}_\nu^\sigma(\alpha_n, \eta_w) = \frac{dec_\nu(\alpha_n, \eta_w)}{\sigma_n(\alpha_n, \eta_w)} = 1,$$

and therefore, using Theorem 8.4.1, we do indeed have

$$\tilde{\psi}_M(\alpha_n \eta_w) = \tilde{\psi}_M(\alpha_n)\tilde{\psi}_M(\eta_w) = \tilde{\psi}_M(\alpha_n)\text{sign}(w)^\Lambda \qquad \text{for some} \quad \Lambda \in \{0, 1\}.$$

Let us now consider statement $(ii)$. By returning to Theorem 4.6.2 and Lemma 6 we see that the cocycles $dec_\nu$ and $\sigma_n$ satisfy,

$$dec_\nu(\eta_w, \alpha_n) = \prod_{(i,j)\in\Phi(w)} (-1)^{\frac{(\rho-1)}{2^r}X_j}(-1)^{\frac{(\rho-1)}{2^r}\max\{X_i, X_j\}}, \tag{8.6}$$

$$\sigma_n(\eta_w, \alpha_n) = \prod_{(i,j)\in\Phi(w)} (-1)^{\frac{(\rho-1)}{2^r}X_i}(-1)^{\frac{(\rho-1)}{2^r}X_iX_j}(a_i^{X_j}a_j^{-X_i}, \pi)_{\nu,m}, \tag{8.7}$$

for each $\alpha_n \in T$ and each $\eta_w \in \mathfrak{M}$.

Now, using equation $(i)$, we must have

$$\tilde{\psi}_M(\eta_w \alpha_n) = \tilde{\psi}_M({}^w\alpha_n \eta_w) = \tilde{\psi}_M({}^w\alpha_n)\tilde{\psi}_M(\eta_w).$$

Therefore, since $\det(\eta_w) = 1$, we may deduce that

$$\begin{aligned} \frac{\tilde{\psi}_M(\alpha_n)}{\tilde{\psi}_M({}^w\alpha_n)} &= \frac{\tilde{\psi}_M(\alpha_n\eta_w)}{\tilde{\psi}_M(\eta_w\alpha_n)} \\ &= \frac{\text{Dec}_\nu^\sigma(\eta_w, \alpha_n)}{\text{Dec}_\nu^\sigma(\alpha_n, \eta_w)} = \frac{dec_\nu(\eta_w, \alpha_n)}{\sigma_n(\eta_w, \alpha_n)}. \end{aligned}$$

Now, by simply substituting equations (8.6) and (8.7) our proof is complete.

□

Finally, it remains to find an expression for the cochains $\tilde{\psi}_M$ when restricted to the torus $T$ of diagonal matrices. We know that, for any cochain $\tilde{\psi}_M$, the restriction $\tilde{\psi}_M|_T$ is a cochain which splits the cocycle $\mathrm{Dec}_\nu^\sigma$ on the torus $T$. Therefore, considering the remark on page 204, we must have

$$\tilde{\psi}_M|_T = \psi_T.\phi,$$

where $\psi_T$ is the cochain found in the previous section and $\phi \in \mathrm{Hom}(T, \mu_m)$.

Thus we are in fact searching for all possible homomorphisms $\phi : T \longmapsto \mu_m$ which satisfy,

$$\tilde{\psi}_M(\alpha_n) = \psi_T(\alpha_n).\phi(\alpha_n) \qquad \alpha_n \in T$$

such that the function $\tilde{\psi}_M$ splits the cocycle $\mathrm{Dec}_\nu^\sigma$ on $M$.

**Theorem 8.4.3** *Any homomorphism $\phi : T \longmapsto \mu_m$ which extends the cochain $\psi_T$ to the monomials $M$ must satisfy,*

$$\phi(\alpha_n) = \varphi(\det(\alpha_n)). \prod_{(i,j)\in\Phi^+} (-1)^{\frac{(\rho-1)}{2^r} X_i},$$

*where $\varphi \in \mathrm{Hom}({k_\nu}^\times, \mu_m)$.*

*Therefore any cochain $\tilde{\psi}_M$ which splits the cocycle* $\mathrm{Dec}_\nu^\sigma$ *over $M \in G_\nu$ must satisfy,*

$$\tilde{\psi}_M(\alpha_n \eta_w) = \mathrm{sign}(w)^\Lambda.\psi_T(\alpha_n).\varphi(\det(\alpha_n)) \prod_{(i,j)\in\Phi^+} (-1)^{\frac{(\rho-1)}{2^r} X_i},$$

*for some homomorphism $\varphi \in \mathrm{Hom}({k_\nu}^\times, \mu_m)$ and some $\Lambda \in \{0,1\}$.*

**PROOF OF THEOREM:**

Since we know that $\tilde{\psi}_M|_T = \psi_T.\phi$, by substituting into equation (*ii*) of Theorem 8.4.2 and then substituting for the function $\psi_T$ as in Theorem 8.3.1, we find

$$\frac{\phi(\alpha_n)}{\phi({}^w\alpha_n)} = \frac{\psi_T({}^w\alpha_n)}{\psi_T(\alpha_n)}. \prod_{(i,j)\in\Phi(w)} (-1)^{\frac{(\rho-1)}{2^r}(X_i+X_j)}(-1)^{\frac{(\rho-1)}{2^r}X_iX_j}(-1)^{\frac{(\rho-1)}{2^r}\max\{X_i,X_j\}}(a_j^{X_i}/a_i^{X_j},\pi)_{\nu,m}$$

$$= \frac{\tau_n({}^w\alpha_n)}{\tau_n(\alpha_n)}.\frac{\prod_{\varsigma\in\Phi^+}(-1)^{\frac{(\rho-1)}{2^r}X_{w^{-1}(i)}X_{w^{-1}(j)}}(a_{w^{-1}(j)}^{X_{w^{-1}(i)}},\pi)_{\nu,m}\prod_{i=1}^n(a_{w^{-1}(i)},\pi)_{\nu,m}^{-\sum_{j=1}^n X_{w^{-1}(j)}/2}}{\prod_{\varsigma\in\Phi^+}(-1)^{\frac{(\rho-1)}{2^r}X_iX_j}(a_j^{X_i},\pi)_{\nu,m}\prod_{i=1}^n(a_i,\pi)_{\nu,m}^{-\sum_{j=1}^n X_j/2}}$$

$$\times \prod_{(i,j)\in\Phi(w)} (-1)^{\frac{(\rho-1)}{2^r}(X_i+X_j)}(-1)^{\frac{(\rho-1)}{2^r}X_iX_j}(-1)^{\frac{(\rho-1)}{2^r}\max\{X_i,X_j\}}(a_j^{X_i}/a_i^{X_j},\pi)_{\nu,m}.$$

Finally, by substituting for the function $\tau_n$, as given in Theorem 4.7.1 on page 92, and tidying the resulting expression we find that the homomorphism $\phi$ must satisfy,

$$\frac{\phi(\alpha_n)}{\phi({}^w\alpha_n)} = \prod_{\varsigma\in\Phi(w)} (-1)^{\frac{(\rho-1)}{2^r}(X_i+X_j)}(a_j^{X_i}/a_i^{X_j},\pi)_{\nu,m}. \prod_{\varsigma\in\Phi^+} (a_{w^{-1}(j)}^{X_{w^{-1}(i)}}/a_j^{X_i},\pi)_{\nu,m}.$$

Furthermore, considering the result given in case (1) of Lemma 8.4.1 we have,

$$\prod_{\varsigma\in\Phi(w)} (a_j^{X_i}/a_i^{X_j},\pi)_{\nu,m} = \prod_{\varsigma\in\Phi^+} (a_j^{X_i}/a_{w^{-1}(j)}^{X_{w^{-1}(i)}},\pi)_{\nu,m}.$$

Therefore we find that the quotient satisfies,

$$\frac{\phi(\alpha_n)}{\phi({}^w\alpha_n)} = \prod_{(i,j)\in\Phi(w)} (-1)^{\frac{(\rho-1)}{2^r}(X_i+X_j)}.$$

Finally, by using the result given in case (2) of Lemma 8.4.1, we see that we may write,

$$\begin{aligned}\frac{\phi(\alpha_n)}{\phi({}^w\alpha_n)} &= \prod_{(i,j)\in\Phi(w)} (-1)^{\frac{(\rho-1)}{2^r}(X_i+X_j)} \\ &= \prod_{(i,j)\in\Phi(w)} \frac{(-1)^{\frac{(\rho-1)}{2^r}X_i}}{(-1)^{\frac{(\rho-1)}{2^r}X_j}} \\ &= \prod_{(i,j)\in\Phi^+} \frac{(-1)^{\frac{(\rho-1)}{2^r}X_i}}{(-1)^{\frac{(\rho-1)}{2^r}X_{w^{-1}(i)}}} \\ &= \frac{\prod_{(i,j)\in\Phi^+} (-1)^{\frac{(\rho-1)}{2^r}X_i}}{\prod_{(i,j)\in\Phi^+}(-1)^{\frac{(\rho-1)}{2^r}X_{w^{-1}(i)}}}.\end{aligned}$$

So, any homomorphism $\phi : T \longmapsto \mu_m$ which extends the cochain $\psi_T$ must be of the form,

$$\phi(\alpha_n) = v(\alpha_n). \prod_{\varsigma\in\Phi^+} (-1)^{\frac{(\rho-1)}{2^r}X_i},$$

where $v : T \longmapsto \mu_m$ is any homomorphism satisfying $v(\alpha_n) = v({}^w\alpha_n)$.

**NB:** Let us be clear that, if in Lemma 8.4.1 we had let $\chi(i,j) = (-1)^{\frac{(\rho-1)}{2^r}X_j}$ we could have found,

$$\phi^\star(\alpha_n) = v(\alpha_n). \prod_{\varsigma\in\Phi^+} (-1)^{\frac{(\rho-1)}{2^r}X_j}.$$

However, when $m$ is odd we have $(-1) = 1$ and therefore we find $\phi^\star(\alpha_n) = \phi(\alpha_n)$.

Let us now suppose that $m$ is even. Then, since we are only considering matrices $\alpha_n \in \mathrm{SL}_n(k_\nu)$ it is easily shown that,

$$\prod_{\varsigma \in \Phi^+} (-1)^{\frac{(\rho-1)}{2^r} X_i} = \prod_{\varsigma \in \Phi^+} (-1)^{\frac{(\rho-1)}{2^r} X_j},$$

and once again we may deduce that $\phi^\star(\alpha_n) = \phi(\alpha_n)$.

Therefore, in both the cases we are considering, the homomorphisms $\phi$ and $\phi^\star$ are identical.

Returning to the proof of our theorem, by referring to Section 8.1.2, we see that any homomorphism $v : T \longmapsto \mu_m$ which satisfies $v(\alpha_n) = v({}^w\alpha_n)$ must be of the form,

$$v(\alpha_n) = \xi_1^{X_1}(a_1, \pi)_{\nu,m}^{\Lambda_1} \ldots \xi_n^{X_n}(a_n, \pi)_{\nu,m}^{\Lambda_n},$$

with $\xi_1 = \ldots = \xi_n \in \mu_m$ and $\Lambda_1 = \ldots = \Lambda_n \in \mathbb{Z}/m$.
However this allows us to write,

$$v(\alpha_n) = \xi_1^{X_1 + \ldots + X_n}(a_1 \ldots a_n, \pi)_{\nu,m}^{\Lambda_1} = \varphi(\det(\alpha_n)),$$

where $\varphi$ is some homomorphism in $\mathrm{Hom}({k_\nu}^\times, \mu_m)$.

In conclusion we have found that the homomorphism $\phi$ must satisfy,

$$\phi(\alpha_n) = \varphi(\det(\alpha_n)). \prod_{\varsigma \in \Phi^+} (-1)^{\frac{(\rho-1)}{2^r} X_i},$$

for some $\varphi \in \mathrm{Hom}({k_\nu}^\times, \mu_m)$.

Therefore, using Theorem 8.4.2, any cochain $\tilde{\psi}_M$ which splits the cocycle $\mathrm{Dec}_\nu^\sigma$ over $M \in G_\nu$ must indeed satisfy,

$$\tilde{\psi}_M(\alpha_n \eta_w) = \mathrm{sign}(w)^\Lambda . \psi_T(\alpha_n) . \varphi(\det(\alpha_n)) \prod_{(i,j) \in \Phi^+} (-1)^{\frac{(\rho-1)}{2^r} X_i},$$

for some $\varphi \in \mathrm{Hom}({k_\nu}^\times, \mu_m)$, $\Lambda \in \{0, 1\}$, which completes the proof of our theorem.

$\square$

**Definition:**

Let us define the cochain $\psi_M$ which splits the cocycle $\mathrm{Dec}_\nu^\sigma$ on $M$ by,

$$\psi_M(\alpha_n \eta_w) := \psi_T(\alpha_n) \prod_{(i,j)\in\Phi^+} (-1)^{\frac{(\rho-1)}{2^r} X_i}.$$

then, as on page 205, Theorem 8.4.3 tells us that any other cochain which splits $\mathrm{Dec}_\nu^\sigma$ does indeed satisfy,

$$\tilde{\psi}_M(\alpha_n\eta_w) = \begin{cases} \varphi(\det(\alpha_n))\psi_M(\alpha_n\eta_w) & \varphi \in \mathrm{Hom}(k_\nu{}^\times, \mu_m), \text{ when } m \text{ is odd} \\ \mathrm{sign}(w)^\Lambda \ \psi_M(\alpha_n\eta_w) & \Lambda \in \{0,1\}, \text{ when } m \text{ is even.} \end{cases}$$

To complete this section let us clarify our results by splitting into the two cases when $m$ is odd and $m$ is even.

**Corollary 12 .**

*Suppose $m$ is odd. Then, any cochain $\tilde{\psi}_M$ which splits the cocycle* $\mathrm{Dec}_\nu^\sigma$ *over $M \in \mathrm{GL}_n(k_\nu)$ must satisfy,*

$$\tilde{\psi}(\mathrm{m}) = \varphi(\det(\alpha_n)) \prod_{\varsigma\in\Phi^+} (a_j^{X_i}, \pi)_{\nu,m} \prod_{i=1}^{n} (a_i, \pi)_{\nu,m}^{-\sum_{j=1}^{n} X_j/2}, \quad \text{some } \varphi \in \mathrm{Hom}(k_\nu{}^\times, \mu_m).$$

*Suppose $m$ is even. Then, any cochain $\tilde{\psi}_M$ which splits the cocycle* $\mathrm{Dec}_\nu^\sigma$ *over* $\mathrm{M} \in \mathrm{SL}_n(k_\nu)$ *must satisfy,*

$$\tilde{\psi}(\mathrm{m}) = \mathrm{sign}(w)^\Lambda . \tau_n(\alpha_n). \prod_{\varsigma\in\Phi^+} (-1)^{\frac{(\rho-1)}{2^r}(X_i + X_i X_j)} (a_j^{X_i}, \pi)_{\nu,m}, \qquad \text{some } \Lambda \in \{0,1\}.$$

## 8.5 The splitting of $\mathrm{Dec}_\nu^\sigma$ on the group $N \subset G_\nu$

In this section we consider the coboundary $\partial\psi_N$ which splits the cocycle $\mathrm{Dec}_\nu^\sigma$ over the group of upper triangular matrices $N \subset G_\nu$. In the cases that $n = 2$ and $n = 3$ we shall be able to completely describe this splitting on the whole of $N$. As we have seen the group $N$ may be generated by,

$$N = \langle n_{i,j}(\pi^{-Z}c) : \ (i,j) \in \Delta \rangle,$$

where the matrices $n_{i,j}(\pi^{-Z}c)$ are as defined in the previous chapters.

Let us also be clear that, since we have $\mathrm{Hom}(N, \mu_m) = 1$, any cochain $\psi_N$ which splits the cocycle $\mathrm{Dec}_\nu^\sigma$ over the group $N \subset G_\nu$ will be unique.

*Using Lemma 7 on page 190 and noting that, for each $n \in N \subset SL_n(k_\nu)$ we have $\det(n) = 1$, we are able to deduce the following:*

**Theorem 8.5.1** *For any matrices $n_1, n_2 \in N$ the cocycle $\mathrm{Dec}_\nu^\sigma$ satisfies,*

$$\mathrm{Dec}_\nu^\sigma(n_1, n_2) = dec_\nu(n_1, n_2).$$

*Using this theorem we are able to deduce that the cocycle $dec_\nu$ itself splits on the group $N$. Therefore, by finding this splitting we shall also be able to find more results concerning the cocycle $dec_\nu$.*

**Theorem 8.5.2** *Let us define $\hat{Z}_{i,j} = \max\{Z_{i,j}, 0\}$ then, the unique cochain $\psi_N$ which splits the cocycle $\mathrm{Dec}_\nu^\sigma$ on the group $N \subset G_\nu$ satisfies,*

$$\psi_N(n_{i,j}(\pi^{-Z_{i,j}}c_{i,j})) = (c_{i,j}, \pi)_{\nu,m}^{\hat{Z}_{i,j}} (-1)^{\frac{(p-1)}{2^r}\frac{\hat{Z}_{i,j}(Z_{i,j}+1)}{2}(j-1)}(-1)^{\frac{(p-1)}{2^r}\frac{\hat{Z}_{i,j}(Z_{i,j}-1)}{2}(n-j)},$$

*for each generator $n_{i,j}(\pi^{-Z_{i,j}}c_{i,j}) \in N$.*

**PROOF OF THEOREM:**

In order to prove this theorem we note that,

$$n_{i,j}(\pi^{-Z_{i,j}}c_{i,j}) = n_{i,j}(\pi^{-Z_{i,j}}c_{i,j}/2).n_{i,j}(\pi^{-Z_{i,j}}c_{i,j}/2).$$

Then, using Theorem 8.5.1, we have

$$\begin{aligned}\frac{\psi_N(n_{i,j}(\pi^{-Z_{i,j}}c_{i,j}/2))\psi_N(n_{i,j}(\pi^{-Z_{i,j}}c_{i,j}/2))}{\psi_N(n_{i,j}(\pi^{-Z_{i,j}}c_{i,j}))} &= \mathrm{Dec}_\nu^\sigma(n_{i,j}(\pi^{-Z_{i,j}}c_{i,j}/2), n_{i,j}(\pi^{-Z_{i,j}}c_{i,j}/2)) \\ &= dec_\nu(n_{i,j}(\pi^{-Z_{i,j}}c_{i,j}/2), n_{i,j}(\pi^{-Z_{i,j}}c_{i,j}/2)).\end{aligned}$$

Using Theorem 6.3.1 on page 149 we see that we have calculated this to be,

$$\frac{\psi_N(n_{i,j}(\pi^{-Z_{i,j}}c_{i,j}/2))^2}{\psi_N(n_{i,j}(\pi^{-Z_{i,j}}c_{i,j}))} = (c_{i,j}/4, \pi)_{\nu,m}^{\hat{Z}_{i,j}}(-1)^{\frac{(p-1)}{2^r}\frac{\hat{Z}_{i,j}(Z_{i,j}+1)}{2}(j-1)}(-1)^{\frac{(p-1)}{2^r}\frac{\hat{Z}_{i,j}(Z_{i,j}-1)}{2}(n-j)}. \tag{8.8}$$

Now, since we know that $\mathrm{Hom}(N, \mu_m) = 1$, if we can find a function $\psi_N$ which satisfies equation (8.8) then this is indeed the unique cochain which splits the cocycle $\mathrm{Dec}_\nu^\sigma$ on N.

Let us now consider the specific case when $m = 2$. Here we find that equation (8.8) simply becomes,

$$\psi_N(n_{i,j}(\pi^{-Z_{i,j}}c_{i,j})) = (c_{i,j}, \pi)_{\nu,m}^{\hat{Z}_{i,j}} (-1)^{\frac{(p-1)}{2^r}\frac{\hat{Z}_{i,j}(Z_{i,j}+1)}{2}(j-1)}(-1)^{\frac{(p-1)}{2^r}\frac{\hat{Z}_{i,j}(Z_{i,j}-1)}{2}(n-j)}.$$

Since this must in fact be the unique cochain which splits $\mathrm{Dec}_\nu^\sigma$ on $N$ over $\mu_2$ we check to see if this result generalizes to any $m$. Then, we do indeed find that,

$$\frac{\psi_N(n_{i,j}(\pi^{-Z_{i,j}}c_{i,j}/2))^2}{\psi_N(n_{i,j}(\pi^{-Z_{i,j}}c_{i,j}))} = \frac{\left((c_{i,j}/2,\pi)_{\nu,m}^{\hat{Z}_{i,j}}(-1)^{\frac{(p-1)}{2^r}\frac{\hat{Z}_{i,j}(Z_{i,j}+1)}{2}(j-1)}(-1)^{\frac{(p-1)}{2^r}\frac{\hat{Z}_{i,j}(Z_{i,j}-1)}{2}(n-j)}\right)^2}{(c_{i,j},\pi)_{\nu,m}^{\hat{Z}_{i,j}}(-1)^{\frac{(p-1)}{2^r}\frac{\hat{Z}_{i,j}(Z_{i,j}+1)}{2}(j-1)}(-1)^{\frac{(p-1)}{2^r}\frac{\hat{Z}_{i,j}(Z_{i,j}-1)}{2}(n-j)}}$$

$$= (c_{i,j}/4,\pi)_{\nu,m}^{\hat{Z}_{i,j}}(-1)^{\frac{(p-1)}{2^r}\frac{\hat{Z}_{i,j}(Z_{i,j}+1)}{2}(j-1)}(-1)^{\frac{(p-1)}{2^r}\frac{\hat{Z}_{i,j}(Z_{i,j}-1)}{2}(n-j)},$$

as required.

□

Now, using the previous theorems we are able to deduce the following:

**Corollary 13** *For any matrices $n \in N$ and $n_{i,j} := n_{i,j}(\pi^{-Z_{i,j}}c_{i,j}) \in N$ with $Z_{i,j} \le 0$ the unique cochain $\psi_N$ satisfies,*

$$\frac{\psi_N(n)\psi_N(n_{i,j})}{\psi_N(n\ n_{i,j})} = dec_\nu(n, n_{i,j}(\pi^{-Z_{i,j}}c_{i,j})) = 1$$

$$\Rightarrow \quad \psi_N(n\ n_{i,j}) = \psi_N(n)\psi_N(n_{i,j}) = \psi_N(n).$$

**PROOF:**

The proof of this follows immediately from Theorems 8.5.1 and 8.5.2.

□

### 8.5.1 The splitting on $N_2$

In this section we shall calculate the cochain $\psi_N$ which splits the cocycle $\mathrm{Dec}_\nu^\sigma$ over the group of upper triangular matrices $N_2 \subset G_\nu \subseteq \mathrm{GL}_2(k_\nu)$. Since $\mathrm{Hom}(N_2, \mu_m) = 1$ the cochain $\psi_N$ will be unique.

**Corollary 14** *The unique cochain $\psi_N$ which splits the cocycle* $\mathrm{Dec}_\nu^\sigma$ *on the group $N_2 \subset G_\nu$ satisfies,*

$$\psi_N\left(\begin{pmatrix} 1 & \pi^{-Z_{1,2}}c_{1,2} \\ 0 & 1 \end{pmatrix}\right) = (c_{1,2},\pi)_{\nu,m}^{\hat{Z}_{1,2}}(-1)^{\frac{(p-1)}{2^r}\frac{\hat{Z}_{1,2}(Z_{1,2}+1)}{2}}.$$

**PROOF:** The proof of this statement follows immediately from Theorem 8.5.2.

□

**Remark:**

Using Theorem 14 it is a simple exercise to confirm that, as in Corollary 4, the cocycle $dec_\nu$ does indeed satisfy,

$$dec_\nu(n_1, n_2) = \mathrm{Dec}_\nu^\sigma(n_1, n_2) = \frac{\psi(n_1)\psi(n_2)}{\psi(n_1 n_2)},$$

with this being a far more convenient way of expressing the result.

### 8.5.2 The splitting on $N_3$

In this section we shall calculate the cochain $\psi_N$ which splits the cocycle $\mathrm{Dec}_\nu^\sigma$ over the group of upper triangular matrices $N_3 \subset G_\nu \subseteq \mathrm{GL}_3(k_\nu)$. Once again, since we have $\mathrm{Hom}(N_3, \mu_m) = 1$, the cochain $\psi_N$ will be unique.

We shall once again use the notation given in Chapter 6 and define,

$$n_{1,2} := \begin{pmatrix} 1 & \pi^{-z_{1,2}}c_{1,2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad n_{1,3} := \begin{pmatrix} 1 & 0 & \pi^{-z_{1,3}}c_{1,3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$n_{2,3} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \pi^{-z_{2,3}}c_{2,3} \\ 0 & 0 & 1 \end{pmatrix}.$$

Then, having defined the matrices $n_{1,2}, n_{2,3}, n_{1,3} \in N$, the general matrix $n \in N$ may be written as,

$$n := \begin{pmatrix} 1 & \pi^{-z_{1,2}}c_{1,2} & \pi^{-z_{1,3}}c_{1,3} \\ 0 & 1 & \pi^{-z_{2,3}}c_{2,3} \\ 0 & 0 & 1 \end{pmatrix} = n_{2,3}n_{1,3}n_{1,2}.$$

Furthermore, the matrices $n_{i,j}$ also satisfy

$$n_{1,2}n_{1,3} = \begin{pmatrix} 1 & \pi^{-z_{1,2}}c_{1,2} & \pi^{-z_{1,3}}c_{1,3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = n_{1,3}n_{1,2}$$

$$n_{2,3}n_{1,3} = \begin{pmatrix} 1 & 0 & \pi^{-z_{1,3}}c_{1,3} \\ 0 & 1 & \pi^{-z_{2,3}}c_{2,3} \\ 0 & 0 & 1 \end{pmatrix} = n_{1,3}n_{2,3}$$

$$n_{2,3}n_{1,2} = \begin{pmatrix} 1 & \pi^{-z_{1,2}}c_{1,2} & 0 \\ 0 & 1 & \pi^{-z_{2,3}}c_{2,3} \\ 0 & 0 & 1 \end{pmatrix}, \quad n_{1,2}n_{2,3} = \begin{pmatrix} 1 & \pi^{-z_{1,2}}c_{1,2} & \pi^{-(z_{1,2}+z_{2,3})}c_{1,2}c_{2,3} \\ 0 & 1 & \pi^{-z_{2,3}}c_{2,3} \\ 0 & 0 & 1 \end{pmatrix}.$$

**Corollary 15** *The unique cochain $\psi_N$ which splits the cocycle* $\mathrm{Dec}_\nu^\sigma$ *on the group* $N_3 \subset G_\nu$ *satisfies,*

$$\psi_N\left(\left(\begin{array}{ccc} 1 & \pi^{-Z_{1,2}}c_{1,2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)\right) = \psi_N(n_{1,2}) = (c_{1,2},\pi)_{\nu,m}^{\hat{Z}_{1,2}} \ (-1)^{\frac{(\rho-1)}{2^r}\hat{Z}_{1,2}}$$

$$\psi_N\left(\left(\begin{array}{ccc} 1 & 0 & \pi^{-Z_{1,3}}c_{1,3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)\right) = \psi_N(n_{1,3}) = (c_{1,3},\pi)_{\nu,m}^{\hat{Z}_{1,3}}$$

$$\psi_N\left(\left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & \pi^{-Z_{2,3}}c_{2,3} \\ 0 & 0 & 1 \end{array}\right)\right) = \psi_N(n_{2,3}) = (c_{2,3},\pi)_{\nu,m}^{\hat{Z}_{2,3}}.$$

**PROOF:**

The proof of this statement follows immediately from Theorem 8.5.2.

□

**Theorem 8.5.3** *The unique cochain $\psi_N$ which splits the cocycle* $\mathrm{Dec}_\nu^\sigma$ *on the group* $N_3 \subset G_\nu$ *satisfies,*

$$\psi_N\left(\left(\begin{array}{ccc} 1 & 0 & \pi^{-Z_{1,3}}c_{1,3} \\ 0 & 1 & \pi^{-Z_{2,3}}c_{2,3} \\ 0 & 0 & 1 \end{array}\right)\right) = \begin{cases} \psi_N(n_{2,3}) = (c_{2,3},\pi)_{\nu,m}^{\hat{Z}_{2,3}} & Z_{1,3} < Z_{2,3} \\ \psi_N(n_{1,3}) = (c_{1,3},\pi)_{\nu,m}^{\hat{Z}_{1,3}} & Z_{2,3} \leq Z_{1,3} \end{cases}$$

$$\psi_N\left(\left(\begin{array}{ccc} 1 & \pi^{-Z_{1,2}}c_{1,2} & \pi^{-Z_{1,3}}c_{1,3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)\right) = \begin{cases} \psi_N(n_{1,2}) = (c_{1,2},\pi)_{\nu,m}^{\hat{Z}_{1,2}} \ (-1)^{\frac{(\rho-1)}{2^r}\hat{Z}_{1,2}} & Z_{1,3} < Z_{1,2} \\ \psi_N(n_{1,3}) = (c_{1,3},\pi)_{\nu,m}^{\hat{Z}_{1,3}} & Z_{1,2} \leq Z_{1,3}. \end{cases}$$

*Furthermore, we also find*

$Z_{1,2} \leq 0$ or $Z_{2,3} \leq 0$:

$$\psi_N\left(\left(\begin{array}{ccc} 1 & \pi^{-Z_{1,2}}c_{1,2} & 0 \\ 0 & 1 & \pi^{-Z_{2,3}}c_{2,3} \\ 0 & 0 & 1 \end{array}\right)\right) = \psi_N(n_{2,3})\psi_N(n_{1,2})$$

$0 < Z_{2,3}, Z_{1,2}$:

$$\psi_N\left(\left(\begin{array}{ccc} 1 & \pi^{-Z_{1,2}}c_{1,2} & 0 \\ 0 & 1 & \pi^{-Z_{2,3}}c_{2,3} \\ 0 & 0 & 1 \end{array}\right)\right) = \begin{array}{c} \psi_N(n_{2,3})\psi_N(n_{1,2}) \ (c_{2,3},\pi)_{\nu,m}^{Z_{1,2}} \\ \times (-1)^{\frac{(\rho-1)}{2^r}\frac{\min\{Z_{1,2},Z_{2,3}\}(\min\{Z_{1,2},Z_{2,3}\}+2Z_{1,2}-1)}{2}} \end{array}.$$

**PROOF OF THEOREM:**

By referring to Theorems 6.4.1, 6.4.2 and 6.4.3 given in Chapter 6 we see that, by using the "hat" notation, the cocycle $dec_\nu$ satisfies,

$$dec_\nu(n_{2,3},n_{1,3}) = \begin{cases} (c_{1,3},\pi)_{\nu,m}^{\hat{Z}_{1,3}} & Z_{1,3} < Z_{2,3} \\ (c_{2,3},\pi)_{\nu,m}^{\hat{Z}_{2,3}} & Z_{2,3} \leq Z_{1,3}. \end{cases}$$

$$dec_\nu(n_{1,3},n_{1,2}) = \begin{cases} (c_{1,3},\pi)_{\nu,m}^{\hat{Z}_{1,3}} & Z_{1,3} < Z_{1,2} \\ (c_{1,2},\pi)_{\nu,m}^{\hat{Z}_{1,2}} (-1)^{\frac{(\rho-1)}{2^r}\hat{Z}_{1,2}} & Z_{1,2} \leq Z_{1,3}. \end{cases}$$

$$dec_\nu(n_{2,3},n_{1,2}) = \begin{cases} 1 & Z_{1,2} \leq 0 \text{ or } Z_{2,3} \leq 0 \\ (c_{2,3},\pi)_{\nu,m}^{-Z_{1,2}}(-1)^{\frac{(\rho-1)}{2^r}\frac{Z_{1,2}(Z_{1,2}+1)}{2}} & 0 < Z_{1,2} < Z_{2,3} \\ (c_{2,3},\pi)_{\nu,m}^{-Z_{1,2}}(-1)^{\frac{(\rho-1)}{2^r}Z_{1,2}Z_{2,3}}(-1)^{\frac{(\rho-1)}{2^r}\frac{Z_{2,3}(Z_{2,3}-1)}{2}} & 0 < Z_{2,3} \leq Z_{1,2}. \end{cases}$$

Now, since we know that,

$$\psi(n_1n_2) = \frac{\psi(n_1)\psi(n_2)}{dec_\nu(n_1,n_2)},$$

using the results given in Theorem 15 together with the previous results for $dec_\nu$ our original statement quickly follows.

□

Finally in order to complete this section we shall consider the cochain $\psi_N$ when applied to the general matrix,

$$n := \begin{pmatrix} 1 & \pi^{-Z_{1,2}}c_{1,2} & \pi^{-Z_{1,3}}c_{1,3} \\ 0 & 1 & \pi^{-Z_{2,3}}c_{2,3} \\ 0 & 0 & 1 \end{pmatrix} = n_{2,3}n_{1,3}n_{1,2}.$$

**Theorem 8.5.4** *Let us define the matrix $n := n_{2,3}n_{1,3}n_{1,2} \in N$ as above. Let us also define,*

$$(1 - c_{2,3}c_{1,2}c_{1,3}^{-1}) = \pi^E e,$$

*for some $E \in \mathbb{Z}^{\geq 0}$ and where either $e = 0$ or $| e |_\nu = 1$.*

*Then, the unique cochain $\psi_N$ which splits the cocycle* $\mathrm{Dec}_\nu^\sigma$ *on the group $N_3 \subset G_\nu$ satisfies,*

- $Z_{1,2} \leq 0$ :
$$\psi_N(n) = \psi_N(n_{2,3}n_{1,3})$$
- $Z_{2,3} < 0$ :
$$\psi_N(n) = \psi_N(n_{1,3}n_{1,2})$$
- $Z_{1,3} < 0$ :
$$\psi_N(n) = \psi_N(n_{2,3}n_{1,2})$$

•$Z_{1,3} < \max\{Z_{2,3}, Z_{1,2}\}$ :

$$\psi_N(n) = \psi_N(n_{2,3}n_{1,2})$$

•$Z_{1,3} \geq \max\{Z_{2,3}, Z_{1,2}\}$ :

∘$Z_{1,2} < Z_{1,3} - Z_{2,3}$ :

$$\psi_N(n) = \psi_N(n_{1,3})$$

∘$Z_{1,2} > Z_{1,3} - Z_{2,3}$ :

$$\psi_N(n) = \psi_N(n_{2,3})\psi_N(n_{1,3})\psi_N(n_{1,2})\,(c_{2,3},\pi)_{\nu,m}^{2Z_{1,2}-Z_{1,3}}(c_{1,3},\pi)_{\nu,m}^{-Z_{1,2}}$$
$$.(-1)^{\frac{(\rho-1)}{2^r}Z_{2,3}Z_{1,3}}(-1)^{\frac{(\rho-1)}{2^r}\frac{Z_{1,2}(Z_{1,2}+1)}{2}}(-1)^{\frac{(\rho-1)}{2^r}\frac{Z_{1,3}(Z_{1,3}+1)}{2}}(-1)^{\frac{(\rho-1)}{2^r}\frac{Z_{2,3}(Z_{2,3}-1)}{2}}$$

∘$Z_{1,2} = Z_{1,3} - Z_{2,3}$ :

$e = 0$ or $E > Z_{2,3}$ :

$$\psi_N(n) = \psi_N(n_{1,3})\psi_N(n_{1,2})\,(c_{1,3},\pi)_{\nu,m}^{-Z_{1,2}}\,(-1)^{\frac{(\rho-1)}{2^r}Z_{1,3}}(-1)^{\frac{(\rho-1)}{2^r}Z_{1,3}Z_{1,2}}$$

$E \leq Z_{2,3}$ :

$$\psi_N(n) = \psi_N(n_{1,3})\,(e,\pi)_{\nu,m}^{Z_{1,2}}\,(-1)^{\frac{(\rho-1)}{2^r}Z_{1,2}E}.$$

**PROOF OF THEOREM:**

In order to prove this theorem we must return to the results given in Theorem 6.4.4 on page 171. Using what we have learnt about the coboundary $\psi_N$ in the preceeding theorems we are now able to re-write these results.

In the case that at least one of the $Z_{i,j} < 0$ we find,

$\bullet Z_{1,2} \leq 0:$ $$dec_\nu(n_{2,3}n_{1,3}, n_{1,2}) = 1$$

$\bullet Z_{2,3} < 0:$ $$dec_\nu(n_{2,3}n_{1,3}, n_{1,2}) = dec_\nu(n_{1,3}, n_{1,2}) = \psi_N(n_{1,3})\psi_N(n_{1,2})\psi_N(n_{1,3}n_{1,2})^{-1}$$

$\bullet Z_{1,3} < 0:$ $$dec_\nu(n_{2,3}n_{1,3}, n_{1,2}) = dec_\nu(n_{2,3}, n_{1,2}) = \psi_N(n_{2,3})\psi_N(n_{1,2})\psi_N(n_{2,3}n_{1,2})^{-1}.$$

Moving onto the cases when $Z_{1,3}, Z_{2,3} \geq 0$ and $Z_{1,2} > 0$ and defining,

$$(1 - c_{2,3}c_{1,2}c_{1,3}^{-1}) = \pi^E e \qquad \text{for some } E \in \mathbb{Z}^{\geq 0} \text{ with } \; e = 0 \text{ or } \mid e \mid_\nu = 1,$$

we are once again able to re-write the results given in Theorem 6.4.4 in terms of the cochain $\psi_N$. That is,

$\bullet\ Z_{1,3} - Z_{2,3} < 0 < Z_{2,3}, Z_{1,2}:$

$dec_\nu(n_{2,3}n_{1,3}, n_{1,2}) = dec_\nu(n_{2,3}, n_{1,2}) = \psi_N(n_{2,3})\psi_N(n_{1,2})\psi_N(n_{2,3}n_{1,2})^{-1}$

$\bullet\ 0 \leq Z_{1,3} - Z_{2,3} \leq Z_{1,3} < Z_{1,2}:$

$dec_\nu(n_{2,3}n_{1,3}, n_{1,2}) = dec_\nu(n_{2,3}, n_{1,2}).\psi_N(n_{2,3})^{-1}\psi_N(n_{1,3}) = \psi_N(n_{1,3})\psi_N(n_{1,2})\psi_N(n_{2,3}n_{1,2})^{-1}$

$\bullet\ 0 < Z_{1,2} < Z_{1,3} - Z_{2,3} \leq Z_{1,3}:$

$dec_\nu(n_{2,3}n_{1,3}, n_{1,2}) = \psi_N(n_{1,2})$

$\bullet\ 0 \leq Z_{1,3} - Z_{2,3} < Z_{1,2} \leq Z_{1,3}:$

$$dec_\nu(n_{2,3}n_{1,3}, n_{1,2}) = \psi_N(n_{2,3})^{-1}\, (c_{2,3}, \pi)_{\nu,m}^{Z_{1,3}-2Z_{1,2}} (c_{1,3}, \pi)_{\nu,m}^{Z_{1,2}}$$
$$(-1)^{\frac{(\rho-1)}{2^r} Z_{2,3}Z_{1,3}} (-1)^{\frac{(\rho-1)}{2^r} \frac{Z_{1,2}(Z_{1,2}+1)}{2}} (-1)^{\frac{(\rho-1)}{2^r} \frac{Z_{1,3}(Z_{1,3}+1)}{2}} (-1)^{\frac{(\rho-1)}{2^r} \frac{Z_{2,3}(Z_{2,3}-1)}{2}}$$

$\bullet\ 0 < Z_{1,2} = Z_{1,3} - Z_{2,3} \leq Z_{1,3}:$

$\circ\ e = 0 \quad \text{or} \quad E > Z_{2,3}:$

$$dec_\nu(n_{2,3}n_{1,3}, n_{1,2}) = (c_{1,3}, \pi)_{\nu,m}^{Z_{1,2}}\, (-1)^{\frac{(\rho-1)}{2^r} Z_{1,3}} (-1)^{\frac{(\rho-1)}{2^r} Z_{1,3}Z_{1,2}}$$

$\circ\ E \leq Z_{2,3}:$

$$dec_\nu(n_{2,3}n_{1,3}, n_{1,2}) = \psi_N(n_{1,2})\, (e, \pi)_{\nu,m}^{-Z_{1,2}}\, (-1)^{\frac{(\rho-1)}{2^r} Z_{1,2}E}.$$

Finally, since we know that the unique cochain which splits $\mathrm{Dec}_\nu^\sigma$ over $N$ satisfies,

$$\psi_N(n_{2,3}n_{1,3}n_{1,2}) = \psi_N(n_{2,3}n_{1,3})\psi_N(n_{1,2})\, dec_\nu(n_{2,3}n_{1,3}, n_{1,2})^{-1}$$
$$= \begin{cases} \psi_N(n_{2,3})\psi_N(n_{1,2})\, dec_\nu(n_{2,3}n_{1,3}, n_{1,2})^{-1} & Z_{1,3} < Z_{2,3} \\ \psi_N(n_{1,3})\psi_N(n_{1,2})\, dec_\nu(n_{2,3}n_{1,3}, n_{1,2})^{-1} & Z_{2,3} \leq Z_{1,3}, \end{cases}$$

putting our results together we are able to conclude that,

$\bullet Z_{1,2} \leq 0:$ $\quad \psi_N(n_{2,3}n_{1,3}, n_{1,2}) = \psi_N(n_{2,3}n_{1,3})$

$\bullet Z_{2,3} < 0:$ $\quad \psi_N(n_{2,3}n_{1,3}, n_{1,2}) = \psi_N(n_{2,3}n_{1,3})\psi_N(n_{1,3})^{-1}\psi_N(n_{1,3}n_{1,2}) = \psi_N(n_{1,3}n_{1,2})$

$\bullet Z_{1,3} < 0:$ $\quad \psi_N(n_{2,3}n_{1,3}, n_{1,2}) = \psi_N(n_{2,3}n_{1,3})\psi_N(n_{2,3})^{-1}\psi_N(n_{2,3}n_{1,2}) = \psi_N(n_{2,3}n_{1,2}).$

$\bullet\ Z_{1,3} - Z_{2,3} < 0 < Z_{2,3}, Z_{1,2}$ and $0 \leq Z_{1,3} - Z_{2,3} \leq Z_{1,3} < Z_{1,2}:$

$$\psi_N(n_{2,3}n_{1,3}, n_{1,2}) = \psi_N(n_{2,3}n_{1,2})$$

$\bullet\ 0 < Z_{1,2} < Z_{1,3} - Z_{2,3} \leq Z_{1,3}:$

$$\psi_N(n_{2,3}n_{1,3}, n_{1,2}) = \psi_N(n_{1,3}),$$

$\bullet\ 0 \leq Z_{1,3} - Z_{2,3} < Z_{1,2} \leq Z_{1,3}:$

$$\psi_N(n_{2,3}n_{1,3}, n_{1,2}) = \psi_N(n_{2,3})\psi_N(n_{1,3})\psi_N(n_{1,2})\ (c_{2,3}, \pi)_{\nu,m}^{2Z_{1,2}-Z_{1,3}}(c_{1,3}, \pi)_{\nu,m}^{-Z_{1,2}}$$
$$(-1)^{\frac{(\rho-1)}{2^r}Z_{2,3}Z_{1,3}}(-1)^{\frac{(\rho-1)}{2^r}\frac{Z_{1,2}(Z_{1,2}+1)}{2}}(-1)^{\frac{(\rho-1)}{2^r}\frac{Z_{1,3}(Z_{1,3}+1)}{2}}(-1)^{\frac{(\rho-1)}{2^r}\frac{Z_{2,3}(Z_{2,3}-1)}{2}}$$

$\bullet\ 0 < Z_{1,2} = Z_{1,3} - Z_{2,3} \leq Z_{1,3}:$

$\circ\ E > Z_{2,3}$ or $e = 0:$

$$\psi_N(n_{2,3}n_{1,3}, n_{1,2}) = \psi_N(n_{1,3})\psi_N(n_{1,2})\ (c_{1,3}, \pi)_{\nu,m}^{-Z_{1,2}}\ (-1)^{\frac{(\rho-1)}{2^r}Z_{1,3}}(-1)^{\frac{(\rho-1)}{2^r}Z_{1,3}Z_{1,2}}$$

$\circ\ E \leq Z_{2,3}:$

$$\psi_N(n_{2,3}n_{1,3}, n_{1,2}) = \psi_N(n_{1,3})\ (e, \pi)_{\nu,m}^{Z_{1,2}}\ (-1)^{\frac{(\rho-1)}{2^r}Z_{1,2}E},$$

which concludes the proof of our theorem.

□

In order to complete this section we shall consider just one further result. This is found immediately as a corollary to the previous theorem.

**Corollary 16** *The unique cochain $\psi_N$ which splits the cocycle $\mathrm{Dec}_\nu^\sigma$ on the group $N_3 \subset G_\nu$ satisfies,*

$\bullet Z_{1,2} < 0$ *or* $Z_{2,3} < 0:$

$$\psi_N(n_{1,2}n_{2,3}) = \psi_N(n_{2,3}n_{1,2}) = \psi_N(n_{2,3})\psi_N(n_{1,2})$$

$\bullet Z_{1,2} \geq 0$ *and* $Z_{2,3} \geq 0:$

$$\psi_N(n_{1,2}n_{2,3}) = \psi_N(n_{2,3})\psi_N(n_{1,2})\ (c_{1,2}, \pi)_{\nu,m}^{Z_{2,3}}(-1)^{\frac{(\rho-1)}{2^r}Z_{2,3}}(-1)^{\frac{(\rho-1)}{2^r}Z_{2,3}Z_{1,2}}.$$

## 8.6 The splitting of $\mathrm{Dec}_\nu^\sigma$ on the full group $G_\nu$

In this section we shall discuss the full splitting of the cocycle $\mathrm{Dec}_\nu^\sigma$ on $G_\nu$. We know that there exists a cochain $\psi$ which satisfies,

$$\mathrm{Dec}_\nu^\sigma(g_1, g_2) = \frac{\psi(g_1)\psi(g_2)}{\psi(g_1 g_2)},$$

for each $g_1,\ g_2 \in G_\nu$.

Since we have also seen that,

$$\mathrm{Hom}(\mathrm{GL}_n(k_\nu), \mu_m) = \mathrm{Hom}(k_\nu{}^\times, \mu_m), \qquad \mathrm{Hom}(\mathrm{SL}_n(k_\nu), \mu_m) = 1,$$

when $m$ is even this cochain will be unique and when $m$ is odd any two cochains will differ by a homomorphism $g \longmapsto \varphi(\det(g))$ for some $\varphi \in \mathrm{Hom}(k_\nu{}^\times, \mu_m)$.

It shall now be convenient for us to look at the two cases, when $m$ is odd and even, separately.

### 8.6.1 The splitting on $\mathrm{GL}_n(k_\nu)$ when $m$ is odd

In this chapter we have found all possible splittings of the cocycle $\mathrm{Dec}_\nu^\sigma$ on both the subgroups $M$ and $N$ in $\mathrm{GL}_n(k_\nu)$. That is, we have found all possible cochains $\tilde{\psi}_M$ and $\psi_N$ which split the cocycle $\mathrm{Dec}_\nu^\sigma$ such that,

$$\mathrm{Dec}_\nu^\sigma|_M = \partial\tilde{\psi}_M \qquad \text{and} \qquad \mathrm{Dec}_\nu^\sigma|_N = \partial\psi_N.$$

The cochains $\tilde{\psi}_M$ and $\psi_N$ have been found to satisfy,

$$\begin{aligned}
\psi_N(n_{i,j}) &= (c_{i,j}, \pi)_{\nu,m}^{\hat{Z}_{i,j}}, \\
\tilde{\psi}_M(\mathrm{m}) &= \varphi(\det(\mathrm{m})) \prod_{\varsigma\in\Phi^+} (a_j^{X_i}, \pi)_{\nu,m} \prod_{i=1}^{n} (a_i, \pi)_{\nu,m}^{-\sum_{j=1}^{n} X_j/2} \\
&= \varphi(\det(\mathrm{m}))\psi_M(\mathrm{m}) \qquad \text{as on page 212} \\
&=: \psi_M^\varphi(\mathrm{m}),
\end{aligned}$$

for some $\varphi \in \mathrm{Hom}(k_\nu{}^\times, \mu_m)$ and where $\hat{Z}_{i,j} = \max\{Z_{i,j}, 0\}$.

Finally, since we have $\mathrm{Hom}(\mathrm{GL}_n(k_\nu), \mu_m) = \mathrm{Hom}(k_\nu{}^\times, \mu_m)$ we may conclude this section by noting the following theorem.

**Theorem 8.6.1** *Any cochain $\psi$ which splits the cocycle $\mathrm{Dec}_\nu^\sigma$ on all of $\mathrm{GL}_n(k_\nu)$ must satisfy,*

$$\psi|_M = \psi_M^\varphi \qquad \text{and} \qquad \psi|_N = \psi_N,$$

*for some $\varphi \in \mathrm{Hom}(k_\nu{}^\times, \mu_m)$.*

### 8.6.2 The splitting on $\mathrm{SL}_n(k_\nu)$ when $m$ is even

In the case that $m$ is even we have found that any two cochains $\tilde{\psi}_M$ and $\psi_N$ which split the cocycle $\mathrm{Dec}_\nu^\sigma$ on $\mathrm{SL}_n(k_\nu)$ such that,

$$\mathrm{Dec}_\nu^\sigma|_M = \partial\tilde{\psi}_M \qquad \text{and} \qquad \mathrm{Dec}_\nu^\sigma|_N = \partial\psi_N,$$

must satisfy,

$$\tilde{\psi}_M(\eta_w) = \mathrm{sign}(w)^\Lambda, \quad \Lambda \in \{0,1\}.$$

$$\begin{aligned}\tilde{\psi}_M(\mathrm{m}) = {} & \mathrm{sign}(w)^\Lambda.\tau_n(\alpha_n).\prod_{\varsigma\in\Phi^+}(-1)^{\frac{(\rho-1)}{2^r}(X_i+X_iX_j)}(a_j^{X_i},\pi)_{\nu,m} \\ & =: \mathrm{sign}(w)^\Lambda.\psi_M(\mathrm{m}) \qquad \text{as on page 212}\end{aligned}$$

$$\psi_N(n_{i,j}) = (c_{i,j},\pi)_{\nu,m}^{\hat{z}_{i,j}}(-1)^{\frac{(\rho-1)}{2^r}\frac{\hat{z}_{i,j}(z_{i,j}+1)}{2}(j-1)}(-1)^{\frac{(\rho-1)}{2^r}\frac{\hat{z}_{i,j}(z_{i,j}-1)}{2}(n-j)}.$$

However, in this case we have $\mathrm{Hom}(\mathrm{SL}_n(k_\nu),\mu_m) = 1$. Therefore when we restrict to $\mathrm{SL}_n(k_\nu)$ the cochain $\psi$ will be unique. This means that we must have either,

$$\psi|_M = \psi_M \qquad \text{or} \qquad \psi|_M = \mathrm{sign}(w)\ \psi_M.$$

In order to discover which of these two cochains is correct we must use the function $\psi_N$. Since we know that $\mathrm{Hom}(N,\mu_m) = 1$ the cochain $\psi_N$, unlike $\psi_M$, must be unique. Therefore we must indeed have,

$$\psi|_N = \psi_N.$$

Now, if we restrict to $\mathrm{SL}_2(k_\nu)$, we see that we are able to write,

$$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}.$$

Extending this to $\mathrm{SL}_n(k_\nu)$ we see that, for any $w_\varsigma = \eta_{s_\varsigma}$ where $\varsigma \in \Delta$ and $s_\varsigma$ is some simple reflection, we find

$$\begin{aligned} w_\varsigma &= n_\varsigma(1).w_\varsigma.n_\varsigma(1).w_\varsigma.n_\varsigma(1) \\ \Rightarrow \qquad \psi(w_\varsigma) &= \psi(nw_\varsigma nw_\varsigma n),\end{aligned}$$

where $n := n_\varsigma(1)$.

Finally, let us be clear that we already know that the cochains satisfy,

$$\tilde{\psi}_M(w_\varsigma) = (-1)^\Lambda, \qquad \text{and} \qquad \psi_N(n_\varsigma(1)) = 1.$$

Therefore, using Theorems 1.4.1, 8.5.2 and Lemmas 6, 7 we are finally able to deduce that the unique cochain $\psi$ satisfies,

$$\begin{aligned}
\psi(w_\varsigma) &= \psi\big(n(w_\varsigma n w_\varsigma)n\big) \\
&= \psi\big(n(w_\varsigma n w_\varsigma)\big)\psi_N(n)\frac{\sigma_n(n(w_\varsigma n w_\varsigma), n)}{dec_\nu(n(w_\varsigma n w_\varsigma), n)} \\
&= \psi\big(n(w_\varsigma n w_\varsigma)\big) && \text{by 1.4.1, 8.5.2, 7} \\
&= \psi_N(n)\psi(w_\varsigma n w_\varsigma)\frac{\sigma_n(n, w_\varsigma n w_\varsigma)}{dec_\nu(n, w_\varsigma n w_\varsigma)} \\
&= \psi(w_\varsigma n w_\varsigma) && \text{by 1.4.1, 7, 8.5.2} \\
&= \psi(w_\varsigma n)\tilde{\psi}_M(w_\varsigma)\frac{\sigma_n(w_\varsigma n, w_\varsigma)}{dec_\nu(w_\varsigma n, w_\varsigma)} \\
&= \psi(w_\varsigma n)(-1)^\Lambda && \text{by 1.4.1, 7, 6} \\
&= \psi_N(n)(-1)^\Lambda\tilde{\psi}_M(w_\varsigma)\frac{\sigma_n(w_\varsigma, n)}{dec_\nu(w_\varsigma, n)} \\
&= \psi_N(n)\big((-1)^\Lambda\big)^2 && \text{by 1.4.1, 7} \\
&= 1 && \text{by 8.5.2 .}
\end{aligned}$$

Finally, we shall state our conclusions in the following theorem:

**Theorem 8.6.2** *The cochain $\psi$ which splits the cocycle $\mathrm{Dec}_\nu^\sigma$ on $\mathrm{SL}_n(k_\nu)$ over $\mu_{2^r}$ satisfies,*

$$\psi|_M = \psi_M \qquad \text{and} \qquad \psi|_N = \psi_N,$$

*where the cochain $\psi_M$ is given by,*

$$\psi_M(\alpha_n\eta_w) = \psi_M(\alpha_n) = \tau_n(\alpha_n)\prod_{\varsigma\in\Phi^+}(-1)^{\frac{(p-1)}{2^r}(X_i+X_iX_j)}(a_j^{X_i},\pi)_{\nu,m}.$$

# Chapter 9

# The cochain $\psi$ on $\mathrm{GL}_2$ and $\mathrm{SL}_2$

## 9.1 Introduction

In this chapter we shall calculate a cochain $\psi$ corresponding to the splitting of the cocycle $\mathrm{Dec}_\nu^\sigma$ on $G_\nu$ over $\mu_m$ where,

$$G_\nu = \begin{cases} \mathrm{GL}_n(k_\nu) & \text{whenever } m \text{ is odd,} \\ \mathrm{SL}_n(k_\nu) & \text{whenever } m \text{ is even.} \end{cases}$$

As we saw in the previous chapter, when $m$ is even, any cochain $\psi$ which splits the cocycle will be unique. However, when $m$ is odd, we find that $\psi$ is not unique and any other cochain $\tilde{\psi}$ which splits the cocycle over $\mathrm{GL}_n(k_\nu)$ must satisfy,

$$\tilde{\psi}(g) = \psi(g)\ \varphi(\det(g)),$$

for some $\varphi \in \mathrm{Hom}({k_\nu}^\times, \mu_m)$.

Throughout this chapter we shall assume it is understood that, when $m$ is odd $(-1)$ and the function "sign" are trivial, while if $m$ is even we have $\det(\alpha_n) = 1$.

In the case that $n = 2$, by explicitly describing this cochain $\psi$, we shall be able to find the isomorphism between the two metaplectic covers of $G_\nu$,

$$\tilde{\mathcal{G}}_{\sigma_2^*} \cong |_{G_\nu}\ \tilde{\mathcal{G}}_{dec_\nu}, \qquad \text{given by,} \qquad (g, \xi) \longmapsto (g, \xi\psi(g)).$$

Using the Bruhat decomposition we have seen that for any matrix $g \in G_\nu$ we may write,

$$g = \begin{pmatrix} \pi^{n_1} g_1 & \pi^{n_{12}} g_{12} \\ \pi^{n_{21}} g_{21} & \pi^{n_2} g_2 \end{pmatrix}$$
$$= \begin{cases} n_1 \alpha_2 & \text{whenever } g_{21} = 0 \\ n_1 \alpha_2 \eta_w n_2 & \text{whenever } g_{21} \neq 0. \end{cases}$$

Therefore, in this chapter we shall calculate all possible values for,

$$\psi(n_1\alpha_2) \quad \text{and} \quad \psi(n_1\alpha_2\eta_w n_2).$$

## 9.2 The function $\psi$

In the previous chapter, for any dimension $n$, we found all possible cochains $\tilde{\psi}_M$ and $\psi_N$ which split the cocycle $\mathrm{Dec}_\nu^\sigma$ on both the monomials and the subgroup of upper triangular matrices in $G_\nu$.

As we saw in Section 8.6, any cochain $\psi$ which splits the cocycle $\mathrm{Dec}_\nu^\sigma$ on all of $G_\nu$ satisfies,

$$\psi|_M = \begin{cases} \psi_M^\varphi \quad \varphi \in \mathrm{Hom}({k_\nu}^\times, \mu_m) & m \text{ is odd} \\ \psi_M & m \text{ is even.} \end{cases}$$
$$\psi|_N = \psi_N.$$

**Definition:**

For the remainder of this work, when we refer to the cochain $\psi$, we shall in fact be considering the function $\psi$ which satisfies,

$$\psi|_M = \begin{cases} \psi_M^1 = \psi_M & \text{whenever } m \text{ is odd} \\ \psi_M & \text{whenever } m \text{ is even}. \end{cases}$$

Let us now return to the results given in the previous chapter and summarise for the case that $n = 2$. Using Theorem 8.4.3 and Corollary 14 we are immediately able to deduce the following:

**Lemma 9.2.1 .**

*For each* $\mathrm{m} = \alpha_2\eta_w \in M$ *the cochain* $\psi$ *which splits the cocycle* $\mathrm{Dec}_\nu^\sigma$ *over* $G_\nu$ *satisfies,*

$$\psi(\mathrm{m}) = \tau_2(\alpha_2).(-1)^{\frac{(\rho-1)}{2^r}(X_1+X_1X_2)}(a_2^{X_1},\pi)_{\nu,m}(a_1a_2,\pi)_{\nu,m}^{-(X_1+X_2)/2}$$

$$= \begin{cases} (a_2^{X_1},\pi)_{\nu,m}(a_1a_2,\pi)_{\nu,m}^{-(X_1+X_2)/2} & \textit{when } m \textit{ is odd,} \\ (-1)^{\frac{(\rho-1)}{2^r}\frac{X_1(|X_1|-1)}{2}}.(a_1^{-X_1},\pi)_{\nu,m} & \textit{when } m \textit{ is even.} \end{cases}$$

*For any* $n = n_{1,2}(\pi^{-Z}c) \in N$ *the cochain* $\psi$ *which splits the cocycle* $\mathrm{Dec}_\nu^\sigma$ *over* $G_\nu \subset \mathrm{GL}_2(k_\nu)$ *satisfies,*

$$\psi_N(n) = \psi_N(\left(\begin{smallmatrix} 1 & \pi^{-Z}c \\ 0 & 1 \end{smallmatrix}\right)) = (c,\pi)_{\nu,m}^{\hat{Z}}\,(-1)^{\frac{(\rho-1)}{2^r}\frac{\hat{Z}(Z+1)}{2}},$$

*where* $\hat{Z} = \max\{Z, 0\}$.

Using Lemma 7 on page 190 and noting that $\det(n) = 1$, we are now able to extend our work from the previous chapter and deduce the following:

**Theorem 9.2.1** *For any matrix* $g \in G_\nu$ *and for each* $n \in N$ *the cocycle* $\mathrm{Dec}_\nu^\sigma$ *satisfies,*

$$\mathrm{Dec}_\nu^\sigma(g,n) = dec_\nu(g,n), \qquad \mathrm{Dec}_\nu^\sigma(n,g) = dec_\nu(n,g).$$

*Therefore, if* $n := n_{1,2}(\pi^{-Z}c) \in N$ *is any matrix with* $Z \le 0$, *the cochain* $\psi$ *must satisfy,*

$$\frac{\psi(g)\psi(n)}{\psi(gn)} = dec_\nu(g, n_{1,2}(\pi^{-Z}c)) = 1 \qquad \textit{by Theorem 5.2.1}$$

$$\Rightarrow \quad \psi(g\ n_{i,j}) = \psi(g)\psi(n) = \psi(g) \qquad \textit{by Lemma 9.2.1.}$$

## 9.3 The full splitting on $\mathrm{GL}_2(k_\nu)$ or $\mathrm{SL}_2(k_\nu)$

In this section we shall use the results given in Chapter 5 to explicitly describe the cochain $\psi$ on the whole of $G_\nu$ in the case that $n = 2$.

Throughout this section we shall once again employ the notation,

$$n_1 := n_{1,2}(\pi^{-Z_1}c_1), \quad n_2 := n_{1,2}(\pi^{-Z_2}c_2) \in N,$$

and then, considering the Bruhat decomposition described earlier, calculate all possible values for,

$$\psi(n_1\alpha_2) \quad \text{and} \quad \psi(n_1\alpha_2\eta_w n_2).$$

Let us also note that since the terms "$(-1)$" are only to be considered in the case when $m$ is even we shall, from this point on, only consider these terms for $\mathrm{SL}_2(k_\nu)$. By this I mean that when considering the exponents of $(-1)$ we shall always let $\alpha_2 \in \mathrm{SL}_2(k_\nu)$ such that $X_2 = -X_1$.

**Theorem 9.3.1** *For each* $n_2 := n_{1,2}(\pi^{-Z_2}c_2) \in N$ *and each* $\alpha_2 \in T \subset G_\nu$ *the cochain* $\psi$ *which splits the cocycle* $\mathrm{Dec}^\sigma_\nu$ *satisfies,*

- $Z_2 \leq 0$ or $0 < Z_2 < (X_1 - X_2, 2X_1)$ :
$$\psi(\alpha_2 n_2) = \psi(\alpha_2)$$

- $Z_2 > 0$ and $(X_1 - X_2, 2X_1) \leq Z_2$ :
$$\psi(\alpha_2 n_2) = \psi(\alpha_2)\psi(n_2)\,.(a_1/a_2, \pi)^{Z_2}_{\nu,m}\;(-1)^{\frac{(\rho-1)}{2^r}\hat{X}_1},$$

*where* $\hat{X}_1 = \max\{X_1, 0\}$.

**PROOF OF THEOREM:**

Since we know that the function $\psi$ must satisfy,

$$\psi(\alpha_2 n_2) = \psi(\alpha_2)\psi(n_2)\; dec_\nu(\alpha_2, n_2)^{-1},$$

the proof of our statement follows directly from the results given in Theorem 5.3.1 on page 99.

□

**Corollary 17** *Let us define* $n_1 := n_{1,2}(\pi^{-Z_1}c_1) \in N$ *then, for each* $\alpha_2 \in T \subset G_\nu$, *we may write*

$$\begin{aligned} n_1\alpha_2 &= n_{1,2}(\pi^{-Z_1}c_1)\alpha_2 \\ &= \alpha_2 n_{1,2}(\pi^{-((X_1-X_2)+Z_1)}a_2c_1/a_1) = \alpha_2\tilde{n_1}. \end{aligned}$$

*Therefore, using the previous theorem, the cochain* $\psi$ *must also satisfy,*

- $Z_1 < 0$ or $Z_1 \leq (X_2 - X_1, -2X_1)$ :
$$\psi(n_1\alpha_2) = \psi(\alpha_2)$$

- $Z_1 \geq 0$ and $(X_2 - X_1, -2X_1) < Z_1$ :
$$\psi(n_1\alpha_2) = \psi(n_1)\psi(\alpha_2)\;(c_1, \pi)^{(X_1-X_2)}_{\nu,m}\;(-1)^{\frac{(\rho-1)}{2^r}(X_1+\hat{X}_1)},$$

*where* $\hat{X}_1 = \max\{X_1, 0\}$.

**Corollary 18** *Since, by Theorem 9.2.1, we must have,*

$$dec_\nu(n_1,\alpha_2) = \frac{\psi(n_1)\psi(\alpha_2)}{\psi(n_1\alpha_2)},$$

*the previous corollary allows us to calculate,*

- $Z_1 < 0$ or $0 \le Z_1 \le (X_2 - X_1, -2X_1)$ :

$$dec_\nu(n_1,\alpha_2) = \psi(n_1) = (c_1,\pi)_{\nu,m}^{\hat{Z}_1}\ (-1)^{\frac{(\rho-1)}{2^r}\frac{\hat{Z}_1(Z_1+1)}{2}}$$

- $(X_2 - X_1, -2X_1) \le 0 \le Z_1$ or $0 \le (X_2 - X_1, -2X_1) < Z_1$ :

$$dec_\nu(n_1,\alpha_2) = (c_1,\pi)_{\nu,m}^{(X_2-X_1)}(-1)^{\frac{(\rho-1)}{2^r}(X_1+\hat{X}_1)},$$

*where* $\hat{X}_1 = \max\{X_1, 0\}$ *and* $\hat{Z}_1 = \max\{Z_1, 0\}$.

**Remark:**

For each $n_1 := n_{1,2}(\pi^{-Z_1}c_1)$, $n_2 := n_{1,2}(\pi^{-Z_2}c_2) \in N$ and each $\alpha_2 \in T \subset G_\nu$ we may write,

$$\psi(n_1\alpha_2 n_2) = \psi(\alpha_2 n_1' n_2) = \psi(\alpha_2 n_2'),$$

where,

$$n_2' = n_{1,2}(\pi^{-((X_1-X_2)+Z_1)}a_2c_1/a_1 + \pi^{-Z_2}c_2).$$

By going through the various possibilities for $n_1$ and $n_2$, and therefore $n_2'$, we may calculate this expression using Theorem 9.3.1. However, since no new understanding of the cochain is gained we shall omit these results.

It is worth pointing out that had we calculated $\psi(n_1\alpha_2 n_2)$ we would have indeed found that Theorems 9.3.1 and 5.5.2 are consistent giving us,

$$\begin{aligned}\psi(n_1\alpha_2 n_2) &= \psi(\alpha_2)\psi(n_2')\, dec_\nu(\alpha_2, n_2')^{-1} \\ &= \psi(n_1\alpha_2)\psi(n_2)\, dec_\nu(n_1\alpha_2, n_2)^{-1} = \psi(n_1\alpha_2 n_2).\end{aligned}$$

**Theorem 9.3.2** *Let* $\eta_w \in \mathfrak{M}$ *where* $w = s_{(1,2)} \in W$. *Then, for each* $\alpha_2 \in G_\nu$ *and each* $n_2 := n_{1,2}(\pi^{-Z_2}c_2) \in N$, *the cochain* $\psi$ *which splits the cocycle* $\mathrm{Dec}_\nu^\sigma$ *on* $G_\nu$ *satisfies,*

- $Z_2 \le 0$ or $0 < Z_2 \le (X_2 - X_1, -2X_1)$ :

$$\psi(\alpha_2\eta_w n_2) = \psi(\alpha_2)$$

- $0 < Z_2$ and $(X_2 - X_1, -2X_1) < Z_2$ :

$$\psi(\alpha_2\eta_w n_2) = \psi(\alpha_2)\psi(n_2)\ .\ (a_2/a_1,\pi)_{\nu,m}^{Z_2}\ (-1)^{\frac{(\rho-1)}{2^r}(X_1+\hat{X}_1)},$$

*where* $\hat{X}_1 = \max\{X_1, 0\}$.

**PROOF OF THEOREM:**

Since we know that the function $\psi$ must satisfy,

$$\psi(\alpha_2\eta_w n_2) = \psi_M(\alpha_2\eta_w)\psi_N(n_2)\; dec_\nu(\alpha_2\eta_w, n_2)^{-1},$$

the proof of our statement follows directly from the results given in Theorem 5.4.1 on page 104.

□

The following results determine $\psi$ on the open Bruhat cell in $\mathrm{GL}_2(k_\nu)$ or $\mathrm{SL}_2(k_\nu)$.

**Corollary 19** *Let $\eta_w \in \mathfrak{M}$ where $w = s_{(1,2)} \in W$. Then, for each $n_2 := n_{1,2}(\pi^{-Z_2}c_2) \in N$ the cochain $\psi$ must satisfy,*

$$\psi(\eta_w n_2) = \psi(n_2). \tag{9.1}$$

*Furthermore, we find that the cochain $\psi$ also satisfies,*

$$\psi(\eta_w n_2 \eta_w) = \psi(n_2).$$

**PROOF:**

The first part is a simple consequence of Theorem 9.3.2. For the second statement we use Lemmas 7, 6 and Theorem 4.2.1 to calculate,

$$\begin{aligned}
\psi(\eta_w n_2 \eta_w) &= \psi(\eta_w n_2)\psi_M(\eta_w)\,\mathrm{Dec}^\sigma_\nu(\eta_w n_2, \eta_w)^{-1} \\
&= \psi(\eta_w n_2)\frac{\sigma_2(\eta_w n_2, \eta_w)}{dec_\nu(\eta_w n_2, \eta_w)} \\
&= \psi(\eta_w n_2).
\end{aligned}$$

Our result then follows from equation (9.1) and Theorem 9.3.2.

□

**Theorem 9.3.3** *Having defined the matrices,*

$$n_1 := n_{1,2}(\pi^{-Z_1}c_1), \quad n_2 := n_{1,2}(\pi^{-Z_2}c_2) \in N,$$

*we let,*

$$(a_2c_1c_2 - a_1) := \pi^H h,$$

*for some $H \geq 0$ and where either $h = 0$ or $|\, h\,|_\nu = 1$.*

*Then, for $\alpha_2 \in T \subset G_\nu$ and $\eta_w \in \mathfrak{M}$, the cochain $\psi$ satisfies,*

• $Z_1 < 0, Z_2 \leq 0$ :

$$\psi(n_1\alpha_2\eta_w n_2) = \psi(\alpha_2)$$

• $Z_1 \geq 0, Z_2 \leq 0$ :

$$\psi(n_1\alpha_2\eta_w n_2) = \psi(n_1\alpha_2\eta_w) = \psi(n_1\alpha_2)$$

• $Z_1 < 0, Z_2 > 0$ :

$$\psi(n_1\alpha_2\eta_w n_2) = \psi(\alpha_2\eta_w n_2).$$

*Furthermore, if we define $\hat{X}_1 = \max\{X_1, 0\}$ and $Z_M = \max\{Z_1, Z_2\}$, then whenever we have both $Z_1,\ Z_2 \geq 0$ the cochain $\psi$ satisfies,*

• $0 \leq Z_1 + Z_2 < (X_2 - X_1, -2X_1)$ :

$$\psi(n_1\alpha_2\eta_w n_2) = \psi(\alpha_2)$$

• $(X_2 - X_1, -2X_1) \leq 0 < Z_1 + Z_2$ or $0 \leq (X_2 - X_1, -2X_1) < Z_1 + Z_2$ :

$$\psi(n_1\alpha_2\eta_w n_2) = \psi(n_1)\psi(\alpha_2)\psi(n_2)\ .(c_1,\pi)_{\nu,m}^{2Z_2-(X_2-X_1)}(a_2/a_1,\pi)_{\nu,m}^{Z_2}(-1)^{\frac{(\rho-1)}{2^r}(X_1+\hat{X}_1)}$$

• $(X_2 - X_1, -2X_1) = Z_1 + Z_2$ :

∘ $h = 0$ or $H > \min\{Z_1, Z_2\}$ :

$$\psi(n_1\alpha_2\eta_w n_2) = \psi(n_1)\psi(\alpha_2)\psi(n_2)\ .(c_1,\pi)_{\nu,m}^{-Z_1}(a_2/a_1,\pi)_{\nu,m}^{Z_2}(-1)^{\frac{(\rho-1)}{2^r}\frac{Z_M(Z_M+1)}{2}}(-1)^{\frac{(\rho-1)}{2^r}Z_M Z_2}$$

∘ $h \neq 0$, $H \leq \min\{Z_1, Z_2\}$ :

$$\begin{aligned}\psi(n_1\alpha_2\eta_w n_2) &= \psi(n_1)\psi(\alpha_2)\psi(n_2)\ .(c_1,\pi)_{\nu,m}^{-Z_1}(a_2/a_1,\pi)_{\nu,m}^{Z_2}(h/a_2c_2,\pi)_{\nu,m}^{Z_2-H}\\ &\quad .(-1)^{\frac{(\rho-1)}{2^r}\frac{Z_1(Z_1+1)}{2}}(-1)^{\frac{(\rho-1)}{2^r}\frac{Z_2(Z_2-1)}{2}}(-1)^{\frac{(\rho-1)}{2^r}\frac{H(H-1)}{2}}(-1)^{\frac{(\rho-1)}{2^r}Z_1 H}.\end{aligned}$$

**PROOF OF THEOREM:**

By using what we already know about the function $\psi$ we are able to write,

$$\begin{aligned}
\psi(n_1\alpha_2\eta_w n_2) &= \frac{\psi(n_1\alpha_2\eta_w)\psi(n_2)}{dec_\nu(n_1\alpha_2\eta_w, n_2)} \\
&= \frac{\psi(n_1\alpha_2)\psi(\eta_w)}{\mathrm{Dec}_\nu^\sigma(n_1\alpha_2, \eta_w)} \cdot \frac{\psi(n_2)}{dec_\nu(n_1\alpha_2\eta_w, n_2)} \\
&= \frac{\psi(n_1\alpha_2)\psi(n_2)}{dec_\nu(n_1\alpha_2\eta_w, n_2)} \\
&= \frac{\psi(n_1)\psi(\alpha_2)\psi(n_2)}{dec_\nu(n_1, \alpha_2)\, dec_\nu(n_1\alpha_2\eta_w, n_2)}.
\end{aligned}$$

However, we have already calculated the values of both $dec_\nu(n_1, \alpha_2)$ and $dec_\nu(n_1\alpha_2\eta_w, n_2)$ in Corollary 18 and Theorem 5.6.2. Therefore, by substituting $\alpha_2 \in \mathrm{SL}_2$ for the exponents of $(-1)$ and then summarising, the proof of our statement quickly follows.

□

**Corollary 20** *Having defined the matrices,*

$$n_1 := n_{1,2}(\pi^{-Z_1}c_1), \quad n_2 := n_{1,2}(\pi^{-Z_2}c_2) \in N,$$

*we find that the cochain $\psi$ satisfies,*

- $Z_1$ or $Z_2 \leq 0$ :

$$\psi(n_1\eta_w n_2) = \psi(n_1)\psi(n_2)$$

- $Z_1 + Z_2 > 0$ :

$$\psi(n_1\eta_w n_2) = \psi(n_1)\psi(n_2)\,.(c_1, \pi)_{\nu,m}^{2Z_2}.$$

## 9.4 General formulae for $\psi$

So far in this chapter we have found the cochain $\psi$, which splits the cocycle $\mathrm{Dec}_\nu^\sigma$ on either $\mathrm{GL}_2(k_\nu)$ or $\mathrm{SL}_2(k_\nu)$, depending on whether $m$ is odd or even. However, as we have seen, this cochain also completely determines the isomorphism,

$$\tilde{\mathcal{G}}_{\sigma_2^*} \cong |_{G_\nu}\tilde{\mathcal{G}}_{dec_\nu}, \qquad \text{given by,} \qquad (g,\xi) \longmapsto (g, \xi\psi(g)),$$

between the two metaplectic covers of $G_\nu$. Considering this isomorphism we would like to be able to be able to find $\psi(g)$ for any $g \in G_\nu$.

Using the Bruhat decomposition we know that any matrix,

$$g = \begin{pmatrix} \pi^{n_1}g_1 & \pi^{n_{12}}g_{12} \\ \pi^{n_{21}}g_{21} & \pi^{n_2}g_2 \end{pmatrix} \in G_\nu,$$

may be written as,

- $g_{21} = 0:$

$$g = \begin{pmatrix} 1 & \pi^{-(n_2-n_{12})}g_{12}/g_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \pi^{n_1}g_1 & 0 \\ 0 & \pi^{n_2}g_2 \end{pmatrix}$$

- $g_{21} \neq 0:$

$$g = \begin{pmatrix} 1 & \pi^{-(n_{21}-n_1)}g_1/g_{21} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (\pi^{(n_1+n_2-n_{21})}g_1g_2/g_{21} - \pi^{n_{12}}g_{12}) & 0 \\ 0 & \pi^{n_{21}}g_{21} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \pi^{-(n_{21}-n_2)}g_2/g_{21} \\ 0 & 1 \end{pmatrix}.$$

Since we have already calculated both the values of,

$$\psi(n_1\alpha_2) \quad \text{and} \quad \psi(n_1\alpha_2\eta_w n_2),$$

by simply re-arranging our results and substituting the constants,

$$c_i, a_i \longmapsto g_i, \qquad X_i, Z_i \longmapsto n_i,$$

we should be able to write down general formulae for the cochain $\psi$ on all of $\mathrm{GL}_2(k_\nu)$ or $\mathrm{SL}_2(k_\nu)$, depending on whether $m$ is odd or even. Since the dependence on $m$ will clearly effect the constants involved we shall, for the remainder of this thesis, consider the two cases separately.

### 9.4.1 $\psi$ on $\mathrm{GL}_2(k_\nu)$ over $\mu_m$ where $m$ is odd

**The case when $g = n_1\alpha_2$**

Here we shall consider all possible matrices $g \in \mathrm{GL}_2(k_\nu)$ such that,

$$g = \begin{pmatrix} \pi^{n_1}g_1 & \pi^{n_{12}}g_{12} \\ 0 & \pi^{n_2}g_2 \end{pmatrix} = \begin{pmatrix} 1 & \pi^{-(n_2-n_{12})}g_{12}/g_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \pi^{n_1}g_1 & 0 \\ 0 & \pi^{n_2}g_2 \end{pmatrix}.$$

Using our previous work in this chapter we may summarise our results as follows:

**Results for $g_{12} = 0$:**

In this case we simply find,

$$\psi\left(\begin{pmatrix} \pi^{n_1}g_1 & 0 \\ 0 & \pi^{n_2}g_2 \end{pmatrix}\right) = (g_2,\pi)_{\nu,m}^{n_1}(g_1g_2,\pi)_{\nu,m}^{-(n_1+n_2)/2}.$$

**Results for $g_{12} \neq 0$:**

For this case we shall have two possibilities. These are,

- $n_2 < n_{12}$ :
  $\psi\left(\begin{pmatrix} \pi^{n_1}g_1 & \pi^{n_{12}}g_{12} \\ 0 & \pi^{n_2}g_2 \end{pmatrix}\right) = (g_2,\pi)_{\nu,m}^{n_1}(g_1g_2,\pi)_{\nu,m}^{-(n_1+n_2)/2}$

- $n_2 \geq n_{12}$ :
  $\psi\left(\begin{pmatrix} \pi^{n_1}g_1 & \pi^{n_{12}}g_{12} \\ 0 & \pi^{n_2}g_2 \end{pmatrix}\right) = (g_2,\pi)_{\nu,m}^{\min\{n_1,n_{12}\}}(g_{12},\pi)_{\nu,m}^{(\max\{n_1,n_{12}\}-n_{12})}(g_1g_2,\pi)_{\nu,m}^{-(n_1+n_2)/2}.$

**The case when $g = n_1\alpha_2\eta_w n_2$**

In this section we shall consider all possible matrices $g$ where $g_{21} \neq 0$. That is, we shall consider matrices,

$$\begin{aligned} g &= \begin{pmatrix} \pi^{n_1}g_1 & \pi^{n_{12}}g_{12} \\ \pi^{n_{21}}g_{21} & \pi^{n_2}g_2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & \pi^{-(n_{21}-n_1)}g_1/g_{21} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (\pi^{(n_1+n_2-n_{21})}g_1g_2/g_{21}-\pi^{n_{12}}g_{12}) & 0 \\ 0 & \pi^{n_{21}}g_{21} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \pi^{-(n_{21}-n_2)}g_2/g_{21} \\ 0 & 1 \end{pmatrix} \\ &= n_1\alpha_2\eta_w n_2. \end{aligned}$$

Before we begin, let us first consider the cochain $\psi$ when applied to the matrices $\alpha_2, \eta_w, n_1, n_2$ which make up the Bruhat decomposition.

Let us begin by considering the value of the cochain $\psi$ on the matrix,

$$\alpha_2 = \begin{pmatrix} (\pi^{(n_1+n_2-n_{21})}g_1g_2/g_{21} - \pi^{n_{12}}g_{12}) & 0 \\ 0 & \pi^{n_{21}}g_{21} \end{pmatrix}.$$

In order to solve this we must consider the following cases:

| | |
|---|---|
| Case (1) | $g_1, g_2 \neq 0, \quad g_{12} = 0$ |
| Case (2) | $g_1, g_2, g_{12} \neq 0, \quad n_1 + n_2 - n_{21} < n_{12}$ |
| Case (3) | $g_1 = g_2 = 0, \quad g_{12} \neq 0$ |
| Case (4) | $g_1 = 0, \quad g_2, g_{12} \neq 0$ |
| Case (5) | $g_2 = 0, \quad g_1, g_{12} \neq 0$ |
| Case (6) | $g_1, g_2, g_{12} \neq 0, \quad n_1 + n_2 - n_{21} > n_{12}$ |
| Case (7) | $g_1, g_2, g_{12} \neq 0, \quad n_1 + n_2 - n_{21} = n_{12}.$ |

In fact these seven distinct cases give us just three results for $\psi(\alpha_2)$. These are found to be,

- Case (1), Case (2) :

$$\psi(g_{(\alpha_2)}) = (g_{21}, \pi)_{\nu,m}^{(n_1+n_2-n_{21})}(g_1g_2, \pi)_{\nu,m}^{-(n_1+n_2)/2}$$

- Case (3), Case (4), Case (5), Case (6) :

$$\psi(g_{(\alpha_2)}) = (g_{21}, \pi)_{\nu,m}^{n_{12}}(g_{12}g_{21}, \pi)_{\nu,m}^{-(n_{12}+n_{21})/2}$$

- Case (7) :

$$\psi(g_{(\alpha_2)}) = (g_{21}, \pi)_{\nu,m}^{n_{12}}((g_1g_2 - g_{12}g_{21}), \pi)_{\nu,m}^{-(n_1+n_2)/2}.$$

As we saw in Lemma 9.2.1 the cochain $\psi$ which splits the cocycle $\mathrm{Dec}_\nu^\sigma$ on $\mathrm{GL}_n(k_\nu)$ satisfies,

$$\psi(\eta_w) = \psi_M(\eta_w) = 1.$$

Finally, considering the value of the cochain $\psi$ on the upper triangular matrices,

$$n_1 := n_{1,2}(\pi^{-(n_{21}-n_1)}g_1/g_{21}), \quad n_2 := n_{1,2}(\pi^{-(n_{21}-n_2)}g_2/g_{21}) \quad \in N,$$

we find that,

$$\psi(n_1) = (g_1/g_{21}, \pi)_{\nu,m}^{\max\{(n_{21}-n_1)\,,\,0\}}$$

$$\psi(n_2) = (g_2/g_{21}, \pi)_{\nu,m}^{\max\{(n_{21}-n_2)\,,\,0\}}.$$

We are now able to consider the cochain $\psi$ applied to any matrix $g \in \mathrm{GL}_n(k_\nu)$ over $\mu_m$ where $m$ is odd. In order to do this we shall once again need to split our results into the various different cases depending on the value of $g \in \mathrm{GL}_n(k_\nu)$.

**Results for $g_1 = g_2 = 0,\ g_{12} \neq 0$:**

In this case we simply find,

$$\psi\Big(\begin{pmatrix} 0 & \pi^{n_{12}}g_{12} \\ \pi^{n_{21}}g_{21} & 0 \end{pmatrix}\Big) = \psi(g_{(\alpha_2)})$$
$$= (g_{21},\pi)_{\nu,m}^{n_{12}}(g_{12}g_{21},\pi)_{\nu,m}^{-(n_{12}+n_{21})/2}.$$

**Results for $g_2 = 0,\ g_1, g_{12} \neq 0$:**

For this case we have,

$$\psi\Big(\begin{pmatrix} \pi^{n_1}g_1 & \pi^{n_{12}}g_{12} \\ \pi^{n_{21}}g_{21} & 0 \end{pmatrix}\Big) = \psi\Big(\begin{pmatrix} 1 & \pi^{-(n_{21}-n_1)}g_1/g_{21} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -\pi^{n_{12}}g_{12} & 0 \\ 0 & \pi^{n_{21}}g_{21} \end{pmatrix}\Big).$$

Therefore, using Corollary 17, we are able to find,

- $n_1 > \min\{n_{12}, n_{21}\}$ :
$$\psi\Big(\begin{pmatrix} \pi^{n_1}g_1 & \pi^{n_{12}}g_{12} \\ \pi^{n_{21}}g_{21} & 0 \end{pmatrix}\Big) = (g_{21},\pi)_{\nu,m}^{n_{12}}(g_{12}g_{21},\pi)_{\nu,m}^{-(n_{12}+n_{21})/2}$$

- $n_1 \leq \min\{n_{12}, n_{21}\}$ :
$$\psi\Big(\begin{pmatrix} \pi^{n_1}g_1 & \pi^{n_{12}}g_{12} \\ \pi^{n_{21}}g_{21} & 0 \end{pmatrix}\Big) = (g_{21},\pi)_{\nu,m}^{n_1}(g_1,\pi)_{\nu,m}^{(n_{12}-n_1)}(g_{12}g_{21},\pi)_{\nu,m}^{-(n_{12}+n_{21})/2}.$$

**Results for $g_1 = 0,\ g_2, g_{12} \neq 0$:**

For this case we have,

$$\psi\Big(\begin{pmatrix} 0 & \pi^{n_{12}}g_{12} \\ \pi^{n_{21}}g_{21} & \pi^{n_2}g_2 \end{pmatrix}\Big) = \psi\Big(\begin{pmatrix} -\pi^{n_{12}}g_{12} & 0 \\ 0 & \pi^{n_{21}}g_{21} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & \pi^{-(n_{21}-n_2)}g_2/g_{21} \\ 0 & 1 \end{pmatrix}\Big).$$

If we now refer to Theorem 9.3.2 on page 228, we find that we have,

- $n_2 \geq \min\{n_{12}, n_{21}\}$ :
$$\psi\Big(\begin{pmatrix} 0 & \pi^{n_{12}}g_{12} \\ \pi^{n_{21}}g_{21} & \pi^{n_2}g_2 \end{pmatrix}\Big) = (g_{21},\pi)_{\nu,m}^{n_{12}}(g_{12}g_{21},\pi)_{\nu,m}^{-(n_{12}+n_{21})/2}$$

- $n_2 < \min\{n_{12}, n_{21}\}$ :
$$\psi\Big(\begin{pmatrix} 0 & \pi^{n_{12}}g_{12} \\ \pi^{n_{21}}g_{21} & \pi^{n_2}g_2 \end{pmatrix}\Big) = (g_{21},\pi)_{\nu,m}^{n_{12}}(g_2/g_{12},\pi)_{\nu,m}^{(n_{21}-n_2)}(g_{12}g_{21},\pi)_{\nu,m}^{-(n_{12}+n_{21})/2}.$$

**Remark:**

The two cases remaining are $g_{12} = 0,\ g_2, g_1 \neq 0$ and $g_{12}, g_2, g_1 \neq 0$. However, we are able to calculate these together since, when $g_{12} = 0$, we shall have,

$$g = \begin{pmatrix} \pi^{n_1}g_1 & \pi^{n_{12}}g_{12} \\ 0 & \pi^{n_2}g_2 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & \pi^{-(n_{21}-n_1)}g_1/g_{21} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \pi^{(n_1+n_2-n_{21})}g_1g_2/g_{21} & 0 \\ 0 & \pi^{n_{21}}g_{21} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & \pi^{-(n_{21}-n_2)}g_2/g_{21} \\ 0 & 1 \end{pmatrix}.$$

Therefore, when calculating the cochain $\psi$ we simply find,

$$\psi(g)\big|_{g_{12}=0} = \psi(g)\big|_{g_{12}\neq 0,\ \ n_1+n_2-n_{21}<n_{12}}.$$

**Results for** $g_{12}=0,\ g_2, g_1 \neq 0$ **and** $g_{12}, g_2, g_1 \neq 0$**:**

$n_{21} < n_1, n_2$**:**

In this case we shall have,

$$\begin{aligned}\psi(g_{(n_1\alpha_2\eta_w n_2)}) &= \psi(g_{(\alpha_2)}),\\ &= \psi\left(\begin{pmatrix}(\pi^{(n_1+n_2-n_{21})}g_1g_2/g_{21}-\pi^{n_{12}}g_{12}) & 0\\ 0 & \pi^{n_{21}}g_{21}\end{pmatrix}\right).\end{aligned}$$

Using the formulae we discovered earlier we are therefore able to write,

- $g_{12}=0$ or $g_{12}\neq 0,\ n_1+n_2-n_{21}<n_{12}$ :
$$\psi(g_{(n_1\alpha_2\eta_w n_2)}) = (g_{21},\pi)_{\nu,m}^{(n_1+n_2-n_{21})}(g_1g_2,\pi)_{\nu,m}^{-(n_1+n_2)/2}$$
- $g_{12}\neq 0,\ n_1+n_2-n_{21}>n_{12}$ :
$$\psi(g_{(n_1\alpha_2\eta_w n_2)}) = (g_{21},\pi)_{\nu,m}^{n_{12}}(g_{12}g_{21},\pi)_{\nu,m}^{-(n_{12}+n_{21})/2}$$
- $g_{12}\neq 0,\ n_1+n_2-n_{21}=n_{12}$ :
$$\psi(g_{(n_1\alpha_2\eta_w n_2)}) = (g_{21},\pi)_{\nu,m}^{n_{12}}((g_1g_2-g_{12}g_{21}),\pi)_{\nu,m}^{-(n_1+n_2)/2}.$$

$n_1 \leq n_{21} \leq n_2$**:**

In this case we have,

$$\begin{aligned}\psi(g_{(n_1\alpha_2\eta_w n_2)}) &= \psi(g_{(n_1\alpha_2)})\\ &= \psi\left(\begin{pmatrix}1 & \pi^{-(n_{21}-n_1)}g_1/g_{21}\\ 0 & 1\end{pmatrix}\begin{pmatrix}(\pi^{(n_1+n_2-n_{21})}g_1g_2/g_{21}-\pi^{n_{12}}g_{12}) & 0\\ 0 & \pi^{n_{21}}g_{21}\end{pmatrix}\right).\end{aligned}$$

Once again using our results from Corollary 17 we are able to find,

- $g_{12}=0$ or $g_{12}\neq 0,\ n_1+n_2-n_{21}<n_{12}$ :
$$\psi(g_{(n_1\alpha_2\eta_w n_2)}) = (g_1,\pi)_{\nu,m}^{(n_2-n_{21})}(g_{21},\pi)_{\nu,m}^{n_1}(g_1g_2,\pi)_{\nu,m}^{-(n_1+n_2)/2}$$
- $g_{12}\neq 0,\ n_1+n_2-n_{21}>n_{12}$ :
$$\psi(g_{(n_1\alpha_2\eta_w n_2)}) = (g_1,\pi)_{\nu,m}^{(n_{12}-\min\{n_1,n_{12}\})}(g_{21},\pi)_{\nu,m}^{\min\{n_1,n_{12}\}}(g_{12}g_{21},\pi)_{\nu,m}^{-(n_{12}+n_{21})/2}$$
- $g_{12}\neq 0,\ n_1+n_2-n_{21}=n_{12}$ :
$$\psi(g_{(n_1\alpha_2\eta_w n_2)}) = (g_1,\pi)_{\nu,m}^{(n_{12}-n_1)}(g_{21},\pi)_{\nu,m}^{n_1}((g_1g_2-g_{12}g_{21}),\pi)_{\nu,m}^{-(n_1+n_2)/2}.$$

$n_2 \leq n_{21} < n_1$:

In this case we have,

$$\begin{aligned}\psi(g_{(n_1\alpha_2\eta_w n_2)}) &= \psi(g_{(\alpha_2\eta_w n_2)})\\ &= \psi\left(\begin{pmatrix} (\pi^{(n_1+n_2-n_{21})}g_1g_2/g_{21}-\pi^{n_{12}}g_{12}) & 0 \\ 0 & \pi^{n_{21}}g_{21}\end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}\begin{pmatrix} 1 & \pi^{-(n_{21}-n_2)}g_2/g_{21} \\ 0 & 1\end{pmatrix}\right).\end{aligned}$$

Using the results given in Theorem 9.3.2 we are now able to write,

- $g_{12}=0$ or $g_{12}\neq 0,\ n_1+n_2-n_{21}<n_{12}$:
$$\psi(g_{(n_1\alpha_2\eta_w n_2)}) = (g_{21},\pi)_{\nu,m}^{n_1}(g_1,\pi)_{\nu,m}^{(n_2-n_{21})}(g_1g_2,\pi)_{\nu,m}^{-(n_1+n_2)/2}$$
- $g_{12}\neq 0,\ n_1+n_2-n_{21}>n_{12}$:
  - $n_{12}\leq n_2$:
$$\psi(g_{(n_1\alpha_2\eta_w n_2)}) = (g_{21},\pi)_{\nu,m}^{n_{12}}(g_{12}g_{21},\pi)_{\nu,m}^{-(n_{12}+n_{21})/2}$$
  - $n_{12}>n_2$:
$$\psi(g_{(n_1\alpha_2\eta_w n_2)}) = (g_{21},\pi)_{\nu,m}^{n_{12}}(g_2/g_{12},\pi)_{\nu,m}^{(n_{21}-n_2)}(g_{12}g_{21},\pi)_{\nu,m}^{-(n_{12}+n_{21})/2}$$
- $g_{12}\neq 0,\ n_1+n_2-n_{21}=n_{12}$:
$$\begin{aligned}\psi(g_{(n_1\alpha_2\eta_w n_2)}) &= (g_{21},\pi)_{\nu,m}^{n_{12}}(g_2,\pi)_{\nu,m}^{(n_{21}-n_2)}\\ &\quad\cdot((g_1g_2/g_{21}-g_{12}),\pi)_{\nu,m}^{(n_2-n_{21})}((g_1g_2-g_{12}g_{21}),\pi)_{\nu,m}^{-(n_{12}+n_{21})/2}.\end{aligned}$$

$n_1, n_2 \leq n_{21}$:

In this case we must consider the full expression for the matrix $g$. That is,

$$\psi(g)=\psi\left(\begin{pmatrix} 1 & \pi^{-(n_{21}-n_1)}g_1/g_{21} \\ 0 & 1\end{pmatrix}\begin{pmatrix} (\pi^{(n_1+n_2-n_{21})}g_1g_2/g_{21}-\pi^{n_{12}}g_{12}) & 0 \\ 0 & \pi^{n_{21}}g_{21}\end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}\begin{pmatrix} 1 & \pi^{-(n_{21}-n_2)}g_2/g_{21} \\ 0 & 1\end{pmatrix}\right)$$

Using the results given in Theorem 9.3.3 the cochain $\psi$ satisfies,

- $g_{12}=0$ or $g_{12}\neq 0,\ n_1+n_2-n_{21}<n_{12}$:
$$\psi(g_{(n_1\alpha_2\eta_w n_2)}) = (g_{21},\pi)_{\nu,m}^{n_1}(g_1,\pi)_{\nu,m}^{(n_2-n_{21})}(g_1g_2,\pi)_{\nu,m}^{-(n_1+n_2)/2}$$
- $g_{12}\neq 0,\ n_1+n_2-n_{21}>n_{12}$:
$$\psi(g_{(n_1\alpha_2\eta_w n_2)}) = (g_{21},\pi)_{\nu,m}^{n_{12}}(g_{12}g_{21},\pi)_{\nu,m}^{-(n_{12}+n_{21})/2}$$
- $g_{12}\neq 0,\ n_1+n_2-n_{21}=n_{12}$:
$$\begin{aligned}\psi(g_{(n_1\alpha_2\eta_w n_2)}) &= (g_{21},\pi)_{\nu,m}^{n_{12}}(g_{12},\pi)_{\nu,m}^{(n_{21}-n_2)}\\ &\quad\cdot((g_1g_2/g_{21}-g_{12}),\pi)_{\nu,m}^{(n_2-n_{21})}((g_1g_2-g_{12}g_{21}),\pi)_{\nu,m}^{-(n_1+n_2)/2}.\end{aligned}$$

### 9.4.2 $\psi$ on $\mathrm{GL}_2(k_\nu)$ over $\mu_m$ where $m$ is even

In the final section of this chapter we shall look specifically at what happens when $m$, the number of roots of unity in $\mu_m$, is even. So, we shall now consider the unique cochain $\psi$ which splits the cocycle $\mathrm{Dec}_\nu^\sigma$ on the whole of $\mathrm{SL}_2(k_\nu)$.

Although we may no longer neglect the terms including $(-1)$ we are able to simplify our equations since our general matrix $g \in \mathrm{SL}_2(k_\nu)$ with $\det(g) = 1$. Therefore, in this case, we are able to use the extra condition,

$$\det(g) = \pi^{(n_1+n_2)} g_1 g_2 - \pi^{(n_{12}+n_{21})} g_{12} g_{21} = 1.$$

**Results for $g = n_1\alpha_2$**

Here we shall consider all possible matrices $g$ such that,

$$\begin{pmatrix} \pi^{n_1} g_1 & \pi^{n_{12}} g_{12} \\ 0 & \pi^{n_2} g_2 \end{pmatrix} = \begin{pmatrix} 1 & \pi^{-(n_2-n_{12})} g_{12}/g_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \pi^{n_1} g_1 & 0 \\ 0 & \pi^{-n_1} g_1^{-1} \end{pmatrix}.$$

Using our previous work in this chapter, we may summarise our results as,

**The cases when $\psi(g_{(n_1\alpha_2)}) = \psi(g_{(\alpha_2)})$:**

**Case (1)** $\quad g_{12} = 0$

**Case (2)** $\quad g_{12} \neq 0, \quad \min\{n_1, n_2\} < n_{12}.$

In each of these cases we simply find,

$$\begin{aligned} \psi(g_{(n_1\alpha_2)}) &= \psi(g_{(\alpha_2)}) \\ &= (g_1, \pi)_{\nu,m}^{-n_1} (-1)^{\frac{(\rho-1)}{m} \frac{n_1(|n_1|-1)}{2}}. \end{aligned}$$

**The case when $\psi(g_{(n_1\alpha_2\eta_w n_2)}) = \psi(g_{(n_1\alpha_2)})$:**

**Case (3)** $\quad g_{12} \neq 0, \quad n_{12} \leq \min\{n_1, n_2\}.$

In this case we use Corollary 17 to calculate,

$$\psi(g_{(n_1\alpha_2)}) = (g_1, \pi)_{\nu,m}^{-n_{12}} (g_{12}, \pi)_{\nu,m}^{(n_1-n_{12})} (-1)^{\frac{(\rho-1)}{2^r} n_1 n_{12}} (-1)^{\frac{(\rho-1)}{2^r} \frac{n_{12}(n_{12}-1)}{2}}.$$

**Results for** $g = n_1\alpha_2\eta_w n_2$

We now consider all matrices $g \in \mathrm{SL}_2(k_\nu)$ which satisfy,

$$g = \begin{pmatrix} \pi^{n_1}g_1 & \pi^{n_{12}}g_{12} \\ \pi^{n_{21}}g_{21} & \pi^{n_2}g_2 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & \pi^{-(n_{21}-n_1)}g_1/g_{21} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \pi^{-n_{21}}g_{21}^{-1} & 0 \\ 0 & \pi^{n_{21}}g_{21} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \pi^{-(n_{21}-n_2)}g_2/g_{21} \\ 0 & 1 \end{pmatrix}.$$

Before we begin this section let us once again consider how the cochain $\psi$ behaves on each of the matrices which make up the Bruhat decomposition for the general matrix in $\mathrm{SL}_n(k_\nu)$.

Considering the value of the cochain $\psi$ on the matrix,

$$\alpha_2 = \begin{pmatrix} \pi^{-n_{21}}g_{21}^{-1} & 0 \\ 0 & \pi^{n_{21}}g_{21} \end{pmatrix},$$

we simply find that,

$$\psi(g_{(\alpha_2)}) = (g_{21}, \pi)_{\nu,m}^{-n_{21}}(-1)^{\frac{(\rho-1)}{2^r}\frac{n_{21}(|n_{21}|-1)}{2}}.$$

As we saw in Lemma 9.2.1 the unique cochain $\psi$ which splits the cocycle $\mathrm{Dec}_\nu^\sigma$ on $\mathrm{SL}_n(k_\nu)$ satisfies,

$$\psi(\eta_w) = \psi_M(\eta_w) = 1.$$

If we now consider the value of the cochain $\psi$ on the upper triangular matrices,

$$n_1 := n_{1,2}(\pi^{-(n_{21}-n_1)}g_1/g_{21}), \quad n_2 := n_{1,2}(\pi^{-(n_{21}-n_2)}g_2/g_{21}) \quad \in N,$$

we find that,

$$\psi(n_1) = (-1)^{\frac{(\rho-1)}{2^r}\frac{\max\{(n_{21}-n_1),0\}((n_{21}-n_1)+1)}{2}}(g_1/g_{21}, \pi)_{\nu,m}^{\max\{(n_{21}-n_1),0\}}$$
$$\psi(n_2) = (-1)^{\frac{(\rho-1)}{2^r}\frac{\max\{(n_{21}-n_2),0\}((n_{21}-n_2)+1)}{2}}(g_2/g_{21}, \pi)_{\nu,m}^{\max\{(n_{21}-n_2),0\}}.$$

We shall now concentrate on finding the value of the cochain $\psi$ when applied to any general matrix $g \in \mathrm{SL}_n(k_\nu)$ over $\mu_m$ where $m$ is even.

**The cases when** $\psi(g_{(n_1\alpha_2\eta_w n_2)}) = \psi(g_{(\alpha_2)})$:

| | |
|---|---|
| **Case (1)** | $g_1 = g_2 = 0$ |
| **Case (2)** | $g_1 = 0, \quad g_2 \neq 0$ and $n_{21} < n_2$ |
| **Case (3)** | $g_1 \neq 0, \quad g_2 = 0$ and $n_{21} < n_1$ |
| **Case (4)** | $g_1, g_2 \neq 0$ and $n_{21} < n_1, n_2$. |

In each of these four cases we simply find,

$$\psi(g_{(n_1\alpha_2\eta_w n_2)}) = \psi\Big(\begin{pmatrix} \pi^{-n_{21}}g_{21}^{-1} & 0 \\ 0 & \pi^{n_{21}}g_{21} \end{pmatrix}\Big) = (g_{21},\pi)_{\nu,m}^{-n_{21}}(-1)^{\frac{(\rho-1)}{2^r}\frac{n_{21}(|n_{21}|-1)}{2}}.$$

**The cases when** $\psi(g_{(n_1\alpha_2\eta_w n_2)}) = \psi(g_{(n_1\alpha_2)})$:

**Case (5)** $\quad g_1 \neq 0,\quad g_2 = 0$ and $n_1 \leq n_{21}$

**Case (6)** $\quad g_1, g_2 \neq 0$ and $n_1 \leq n_{21} \leq n_2$.

In each of these cases, using the results given in Corollary 17, we find

$$\begin{aligned}\psi(g_{(n_1\alpha_2\eta_w n_2)}) &= \psi\Big(\begin{pmatrix} 1 & \pi^{-(n_{21}-n_1)}g_1/g_{21} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \pi^{-n_{21}}g_{21}^{-1} & 0 \\ 0 & \pi^{n_{21}}g_{21} \end{pmatrix}\Big) \\ &= (g_1,\pi)_{\nu,m}^{-(n_{21}+n_1)}(g_{21},\pi)_{\nu,m}^{n_1}(-1)^{\frac{(\rho-1)}{2^r}n_{21}n_1}(-1)^{\frac{(\rho-1)}{2^r}\frac{n_1(n_1-1)}{2}}.\end{aligned}$$

**The cases when** $\psi(g_{(n_1\alpha_2\eta_w n_2)}) = \psi(g_{(\alpha_2\eta_w n_2)})$:

**Case (7)** $\quad g_1 = 0,\quad g_2 \neq 0$ and $n_2 \leq n_{21}$

**Case (8)** $\quad g_1, g_2 \neq 0$ and $n_2 \leq n_{21} < n_1$.

For these cases we have,

$$\psi(g_{(n_1\alpha_2\eta_w n_2)}) = \psi\Big(\begin{pmatrix} \pi^{-n_{21}}g_{21}^{-1} & 0 \\ 0 & \pi^{n_{21}}g_{21} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & \pi^{-(n_{21}-n_2)}g_2/g_{21} \\ 0 & 1 \end{pmatrix}\Big).$$

Therefore, using the results given in Theorem 9.3.2, we find

- $-n_2 \leq n_{21}$ :

$$\psi(g_{(\alpha_2\eta_w n_2)}) = (g_{21},\pi)_{\nu,m}^{-n_{21}}(-1)^{\frac{(\rho-1)}{2^r}\frac{n_{21}(n_{21}-1)}{2}}$$

- $-n_2 > n_{21}$ :

$$\psi(g_{(\alpha_2\eta_w n_2)}) = (g_{21},\pi)_{\nu,m}^{-n_2}(g_2,\pi)_{\nu,m}^{(n_{21}-n_2)}(-1)^{\frac{(\rho-1)}{2^r}n_{21}n_2}(-1)^{\frac{(\rho-1)}{2^r}\frac{n_2(n_2-1)}{2}}.$$

**The case when** $\psi(g) = \psi(g_{(n_1\alpha_2\eta_w n_2)})$:

**Case (9)** $\quad g_1, g_2 \neq 0$ and $n_1, n_2 \leq n_{21}$.

In this case we must consider the full expression,

$$\psi(g_{(n_1\alpha_2\eta_w n_2)}) = \psi\Big(\begin{pmatrix} 1 & \pi^{-(n_{21}-n_1)}g_1/g_{21} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \pi^{-n_{21}}g_{21}^{-1} & 0 \\ 0 & \pi^{n_{21}}g_{21} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & \pi^{-(n_{21}-n_2)}g_2/g_{21} \\ 0 & 1 \end{pmatrix}\Big).$$

Using the results given in Theorem 9.3.3 we find that the cochain $\psi$ satisfies,

• $n_1 + n_2 > 0$ :

$$\psi(n_1\alpha_2\eta_w n_2) = (g_{21},\pi)_{\nu,m}^{-n_{21}}(-1)^{\frac{(\rho-1)}{2^r}\frac{n_{21}(n_{21}-1)}{2}}$$

• $n_1 + n_2 < 0$

$$\psi(n_1\alpha_2\eta_w n_2) = (g_2,\pi)_{\nu,m}^{(n_{21}-n_2)}(g_1,\pi)_{\nu,m}^{(n_{21}-n_1-2n_2)}(g_{21},\pi)_{\nu,m}^{(n_1+n_2-n_{21})}$$
$$.(-1)^{\frac{(\rho-1)}{2^r}n_{21}(n_1+n_2)}(-1)^{\frac{(\rho-1)}{2^r}\frac{n_{21}(n_{21}+1)}{2}}(-1)^{\frac{(\rho-1)}{2^r}\frac{n_1(n_1-1)}{2}}(-1)^{\frac{(\rho-1)}{2^r}\frac{n_2(n_2-1)}{2}}$$

• $n_1 + n_2 = 0$ :

∘ $g_{12} = 0$ or $g_{12} \neq 0,\ n_{12} > \min\{n_1, n_2\}$ :

$$\psi(n_1\alpha_2\eta_w n_2) = (g_2,\pi)_{\nu,m}^{(n_{21}+n_1)}(g_{21},\pi)_{\nu,m}^{n_1}(-1)^{\frac{(\rho-1)}{2^r}n_{21}n_1}(-1)^{\frac{(\rho-1)}{2^r}\frac{n_1(|n_1|+1)}{2}}$$

∘ $g_{12} \neq 0,\ n_{12} \leq \min\{n_1, n_2\}$ :

$$\psi(n_1\alpha_2\eta_w n_2) = (g_2,\pi)_{\nu,m}^{(n_{12}+n_{21})}(g_{21},\pi)_{\nu,m}^{n_1}(g_{12},\pi)_{\nu,m}^{(n_1-n_{12})}$$
$$.(-1)^{\frac{(\rho-1)}{2^r}(n_1+n_1n_{12}+n_1n_{21})}(-1)^{\frac{(\rho-1)}{2^r}\frac{n_{12}(n_{12}-1)}{2}}.$$

# Bibliography


[1] W. D. Banks, J. L. Levy and M. R. Sepanski : *Block-compatible Metaplectic Cocycles* J. reine angew. Math. 507, 131-163 (1999)

[2] R. W. Carter *Some Aspects of the Representation Theory of Finite Groups of Lie Type* Mathematical Institute, Oxford (1977) 6-35.

[3] R. Hill : *Space Forms and Higher Metaplectic Groups* Math. Ann. 310, 753-775 (1998)

[4] R. Hill : *Metaplectic covers of* $\mathrm{GL}_n$ and the Gauss-Schering Lemma 21st Journées Arithmétiques (Rome, 2001). J. Théor. Nombres Bordeaux 13 (2001), no. 1, 189-199.

[5] T. Kubota : *On Automorphic Functions and the Reciprocity Law in a Number Field* Lecture notes vol. 2, Kyoto Univ. (1969)

[6] T. Kubota : *Topological Covering of* SL*(2) over a Local Field* J. Math. Soc. Japan 19, 114-121 (1967)

[7] H. Matsumoto : *Sur les sous-groupe arithmétiques des gropes semi-simples déployés* Ann. Sci. Éc. Norm. Sup. (4) 2, 1-62 (1969)

[8] R. Steinberg : *Lectures on Chevalley Groups* Yale Math. Dept. (1968)